Elaborations Example Exam 2 Wiskunde B 2018

Question 1a – 4 points

For $x \neq 0$ we have:

$$f(x) = \frac{-2+2\sqrt{x+1}}{x} \cdot \frac{1+\sqrt{x+1}}{1+\sqrt{x+1}} = \frac{2(-1+\sqrt{x+1})(1+\sqrt{x+1})}{x\left(1+\sqrt{x+1}\right)} = \frac{2\left(-1^2+\left(\sqrt{x+1}\right)^2\right)}{x\left(1+\sqrt{x+1}\right)}$$

$$= \frac{2(-1+x+1)}{x\left(1+\sqrt{x+1}\right)} = \frac{2x}{x\left(1+\sqrt{x+1}\right)} = \frac{2}{1+\sqrt{x+1}} = h(x)$$

Alternative:

$$f(x) = h(x) \Leftrightarrow \frac{-2+2\sqrt{x+1}}{x} = \frac{2}{1+\sqrt{x+1}} \Leftrightarrow \frac{2(-1+\sqrt{x+1})(1+\sqrt{x+1})}{x} = 2 \Leftrightarrow \frac{2x}{x} = 2$$

Question 1b – 4 points

For $x \neq 0$ we have:

$$g(x) = \frac{4x^2+x}{x} = 4x+1$$

k: $y = 4x+1$ with $x = 0$ yields $y = 1$, and also $h(0) = 1$.

The perforation of both f and g is therefore point (0,1).

$$h'(x) = \frac{-\frac{2}{2\sqrt{x+1}}}{\left(1+\sqrt{x+1}\right)^2} = -\frac{1}{\sqrt{x+1}\cdot\left(1+\sqrt{x+1}\right)^2}$$

The slope of *k* is 4; the slope of h in (0,1) is $h'(0) = -\frac{1}{4}$

The product of these slopes is -1, so the graphs are perpendicular in their intersection.

Question 2a – 4 points

$f(p) - g(p) = \ln(4) \Leftrightarrow \ln(9-2p) - \ln(3-p) = \ln(4) \Leftrightarrow \ln\left(\frac{9-2p}{3-p}\right) = \ln(4)$

$$\Leftrightarrow \frac{9-2p}{3-p} = 4 \Leftrightarrow 9-2p = 4(3-p) \Leftrightarrow 9-2p = 12-4p \Leftrightarrow 2p = 3 \Leftrightarrow p = \tfrac{3}{2}$$

$g(p) - f(p) = \ln(4)$ has no solutions as you can see in the graph.

Question 2b – 7 points

$f'(x) = -\frac{2}{9-2x}$; $g'(x) = -\frac{1}{3-x}$

$f(0) = \ln(9)$; $f'(0) = -\frac{2}{9}$; $g(0) = \ln(3)$; $g'(0) = -\frac{1}{3}$

Therefore, the tangent lines are $y = -\frac{2}{9}x + \ln(9)$ and $y = -\frac{1}{3}x + \ln(3)$

Now solve: $-\frac{2}{9}x + \ln(9) = -\frac{1}{3}x + \ln(3)$

This yields $\frac{1}{9}x = \ln(3) - \ln(9) \Leftrightarrow x = 9(\ln(3) - \ln(9))$

So $x = 9\ln\left(\frac{3}{9}\right) = 9\ln\left(\frac{1}{3}\right) = \ln(3^{-9}) = -9\ln(3)$

The second, third and fourth expression are all OK.

Question 2c – 5 points

$f(x) = h(x) \Leftrightarrow \ln(9-2x) = 2\ln(x+3) \Leftrightarrow \ln(9-2x) = \ln((x+3)^2) \Leftrightarrow 9-2x = x^2+6x+9$

This yields $x^2 + 8x = 0 \Leftrightarrow x = 0 \vee x = -8$

$x = -8$ does not suffice; $x = 0$ yields intersection $(0, \ln(9))$

Question 2d – 7 points

$y = \ln(3-x) \Leftrightarrow \mathrm{e}^y = 3-x \Leftrightarrow x = 3 - \mathrm{e}^y$

Therefore we have to compute $\pi \cdot \int_0^{\ln(3)} (3-\mathrm{e}^y)^2 \, \mathrm{d}y$

An antiderivative of $(3-\mathrm{e}^y)^2 = 9 - 6\mathrm{e}^y + \mathrm{e}^{2y}$ is $G(y) = 9y - 6\mathrm{e}^y + \frac{1}{2}\mathrm{e}^{2y}$

So the volume is $\pi \cdot \left(9\ln(3) - 6 \cdot 3 + \frac{1}{2} \cdot 9 - \left(0 - 6 + \frac{1}{2}\right)\right) = \pi(9\ln(3) - 8)$

Question 3a – 6 points

$\angle CBP = 90°$, so our friend Thales states that the centre of the circle through *B*, *C* and *P* is de midpoint of the hypothenuse *PC* of triangle *BPC*. This is the point $\left(-1, -\frac{1}{2}\right)$

Therefore, the equation of the circle has the form $(x+1)^2 + \left(y + \frac{1}{2}\right)^2 = r^2$

Substitution of the coordinates of *B* yields $r^2 = (0+1)^2 + \left(4 + \frac{1}{2}\right)^2 = 1 + 20\frac{1}{4} = 21\frac{1}{4}$
You could also enter the coordinates of C or P.

Alternative 1:

The line segment bisector of *B* and *C* passes through $(-2,0)$ and has slope $-\frac{1}{2}$
Since it is parallel to line segment AB.

Therefore, the equation of this line is $y = -\frac{1}{2}(x+2) \Leftrightarrow y = -\frac{1}{2}x - 1$

The line segment bisector of *B* and *P* passes through $(1,13\frac{1}{2})$ and has slope 2
Since it is parallel to line segment BC.

Therefore, the equation of this line is $y - 3\frac{1}{2} = 2(x-1) \Leftrightarrow y = 2x + 1\frac{1}{2}$

The centre of the circle is the intersection of these lines. This is found by solving

$-\frac{1}{2}x - 1 = 2x + 1\frac{1}{2} \Leftrightarrow -2\frac{1}{2}x = 2\frac{1}{2} \Leftrightarrow x = -1$, dus $y = -\frac{1}{2}$

Computation of the equation of the circle as above..

Alternative 2:

Substitution of the coordinates of $B(0,4)$, $C(-4,-4)$ and $P(2,3)$ in $(x-a)^2 + (y-b)^2 = r^2$ yields three equations in three unknowns from which a, b and r^2 can be solved.

Question 3b – 6 points

The straight line through $P(2,3)$ and $D(4,-8)$ has slope $-\frac{11}{2}$.

Therefore, the equation of this line is $y - 3 = -\frac{11}{2}(x-2) \Leftrightarrow y = -\frac{11}{2}x + 14$

The line through *C* and *Q* is perpendicular to this line, so its slope is $\frac{2}{11}$.

The equation of the line trough $C(-4,-4)$ and *Q* is therefore $y + 4 = \frac{2}{11}(x+4) \Leftrightarrow y = \frac{2}{11}x - \frac{36}{11}$

Q is the intersection of these lines, so solve::

$-\frac{11}{2}x + 14 = \frac{2}{11}x - \frac{36}{11} \Leftrightarrow -121x + 308 = 4x - 72 \Leftrightarrow -125x = -380 \Leftrightarrow x = \frac{76}{25}$

This yields $y = -\frac{11}{2} \cdot \frac{76}{25} + 14 = -\frac{836}{50} + \frac{700}{50} = -\frac{136}{50} = -\frac{68}{25} \left(= \frac{2}{11} \cdot \frac{76}{25} - \frac{36}{11} = \frac{152}{275} - \frac{900}{275} = -\frac{748}{275} = -\frac{68}{25}\right)$

Alternatives:

Thales states that Q is on the circle from question a, so find the intersection of this circle with line segment PD. There are also several elaborations using vectors.

Question 3c – 6 points

The projection of *P* on side *CD* is *S* and the projection of *Q* on side *CD* is *T*.

Note that triangles *PSD* and *QTD* are similar.

The area of triangle *CQD* is therefore $\frac{1}{2}|CD| \cdot |QT|$

This must be equal to $\frac{1}{3} \cdot |CD|^2$

This yields $|QT| = \frac{2}{3}|CD|$

Because of the similarity mentioned above, this yields $|DQ| = \frac{2}{3}|DP|$.

So we get: $\overrightarrow{OQ} = \overrightarrow{OD} + \overrightarrow{DQ} = \overrightarrow{OD} + \frac{2}{3} \cdot \overrightarrow{DP} = \begin{pmatrix} 4 \\ -8 \end{pmatrix} + \frac{2}{3} \cdot \begin{pmatrix} -2 \\ 11 \end{pmatrix} = \begin{pmatrix} 2\frac{2}{3} \\ -\frac{2}{3} \end{pmatrix}$, thus $x_Q = 2\frac{2}{3}$ and $y_Q = -\frac{2}{3}$

Question 4a – 5 points

$f_1'(x) = 2x \cdot \mathrm{e}^x + x^2 \cdot \mathrm{e}^x$

$f_1''(x) = 2 \cdot \mathrm{e}^x + 2x \cdot \mathrm{e}^x + 2x \cdot \mathrm{e}^x + x^2$

$f_1''(x) = 0 \Leftrightarrow x^2 + 4x + 2 = 0 \Leftrightarrow x = \dfrac{-4 + \sqrt{8}}{2} = -2 + \sqrt{2} \vee x = \dfrac{-4 - \sqrt{8}}{2} = -2 - \sqrt{2}$

Question 4b – 6 points

$f_a'(x) = 2x \cdot \mathrm{e}^{ax} + x^2 \cdot a\mathrm{e}^{ax}$

$f_a'(x) = 0 \Leftrightarrow ax^2 + 2x = 0 \Leftrightarrow x = 0 \vee x = -\dfrac{2}{a}$

So in the maximum we have $x = -\frac{2}{a} \Leftrightarrow a = -\frac{2}{x}$

Substitution of $a = -\frac{2}{x}$ in $y = x^2 \cdot \mathrm{e}^{ax}$ yields $y = x^2 \cdot \mathrm{e}^{-2}$

Question 4c – 7 points

In a point where the graph of f_2 and the line $y = px$ are touching, we have: $f_2(x) = px$ and $f_2'(x) = p$

This yields $x^2\mathrm{e}^{2x} = px$ and $2x\mathrm{e}^{2x} + 2x^2\mathrm{e}^{2x} = p$

Combining this yields: $x^2\mathrm{e}^{2x} = (2x\mathrm{e}^{2x} + 2x^2\mathrm{e}^{2x}) \cdot x \Leftrightarrow x^2\mathrm{e}^{2x} = x^2\mathrm{e}^{2x}(2 + 2x) \Leftrightarrow x = 0 \vee x = -\frac{1}{2}$

$x = 0$ yields the tangent line $y = 0$ (the *x*-axis).

$x = -\frac{1}{2}$ yields $p = -\mathrm{e}^{-1} + \frac{1}{2}\mathrm{e}^{-1} = -\frac{1}{2}\mathrm{e}^{-1}$, so the tangent line is $y = -\frac{1}{2}\mathrm{e}^{-1} \cdot x$

Question 5a – 4 points

On a vertical line, the *x*-coordinates are equal, so we must have: $x(\pi - a) = x(a)$, and that is true:
$x(\pi - a) = \cos(\pi - a)\sin(2\pi - 2a) = -\cos(a)\sin(-2a) = -\cos(a) \cdot -\sin(2a) = \cos(a)\sin(2a) = x(a)$

Question 5b – 5 points

$d(P_t, x-as) = |y(t)|$; $d(P_t, y-as) = |x(t)|$. Therefore, we must have $|y(t)| = 2|x(t)|$.

This yields: $|\cos(t)| = |2\cos(t)\sin(2t)| \Leftrightarrow |2\sin(2t)| = 1$
The points where $\cos(t) = 0$ *are disregarded.*

$2\sin(2t) = 1 \Leftrightarrow \sin(2t) = \frac{1}{2} \Leftrightarrow 2t = \frac{1}{6}\pi + k \cdot 2\pi \vee 2t = \frac{5}{6}\pi + k \cdot 2\pi \Leftrightarrow t = \frac{1}{12}\pi + k \cdot \pi \vee t = \frac{5}{12}\pi + k \cdot \pi$

$2\sin(2t) = -1$

$\Leftrightarrow \sin(2t) = -\frac{1}{2} \Leftrightarrow 2t = -\frac{1}{6}\pi + k \cdot 2\pi \vee 2t = -\frac{5}{6}\pi + k \cdot 2\pi \Leftrightarrow t = -\frac{1}{12}\pi + k \cdot \pi \vee t = -\frac{5}{12}\pi + k \cdot \pi$

Fourth time: $t = -\frac{1}{12}\pi + \pi = \frac{11}{12}\pi$
First time $\frac{1}{12}\pi$*; second time* $\frac{5}{12}\pi$*; third time* $-\frac{5}{12}\pi + \pi = \frac{7}{12}\pi$

Question 5c – 5 points

$$\overrightarrow{OP_t} = \begin{pmatrix} x\left(\frac{3}{4}\pi\right) \\ y\left(\frac{3}{4}\pi\right) \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{3}{4}\pi\right) \cdot \sin\left(1\frac{1}{2}\pi\right) \\ \cos\left(\frac{3}{4}\pi\right) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} \end{pmatrix}$$

$x'(t) = -\sin(t) \cdot \sin(2t) + \cos(t) \cdot 2\cos(2t)$, so $x'\left(\frac{3}{4}\pi\right) = -\frac{1}{2}\sqrt{2} \cdot -1 + 0 = \frac{1}{2}\sqrt{2}$

$y'(t) = -\sin(t)$, so $y'\left(\frac{3}{4}\pi\right) = -\frac{1}{2}\sqrt{2}$

Therefore, we get $\vec{v} = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} \end{pmatrix} = \overrightarrow{OP_t}$

Question 6a – 5 points

$$g_p(x) = \frac{x^2 - 5x + 6}{2x - p} \Rightarrow g_p'(x) = \frac{(2x-5)(2x-p) - (x^2 - 5x + 6) \cdot 2}{(2x-p)^2} \Rightarrow g_p'(2) = \frac{-(4-p) - 0}{(4-p)^2} = \frac{1}{p-4}$$

The slope of $y = \frac{1}{4}x^2 - 1$ in (2,0) is $\frac{1}{4} \cdot 2 \cdot 2 = 1$

So we must have $\frac{1}{p-4} = 1 \Leftrightarrow p - 4 = 1 \Leftrightarrow p = 5$

Question 6b – 5 points

$$g_8(x) = \frac{x^2 - 5x + 6}{2x - 8}$$

$2x - 8 = 0 \Leftrightarrow x = 4$; for $x = 4$, the numerator is not equal to 0, so the vertical asymptote is $x = 4$.

$$\frac{x^2 - 5x + 6}{2x - 8} = \frac{x^2 - 4x - x + 6}{2x - 8} = \frac{x^2 - 4x}{2x - 8} + \frac{-x + 4}{2x - 8} + \frac{2}{2x - 8} = \tfrac{1}{2}x - \tfrac{1}{2} + \frac{2}{2x - 8}$$

So the oblique asymptote is $y = \frac{1}{2}x - \frac{1}{2}$.

Extra item:

For $x \neq 2$ we have $g_4(x) = \frac{(x-2)(x-3)}{(2x-4)} = \frac{1}{2}(x-3)$.

Therefore, the graph is the line $y = \frac{1}{2}(x-3)$ with a perforation in $\left(2, -\frac{1}{2}\right)$.

For $x \neq 3$ we have $g_6(x) = \frac{(x-2)(x-3)}{(2x-6)} = \frac{1}{2}(x-2)$.

Therefore, the graph is the line $y = \frac{1}{2}(x-2)$ with a perforation in $\left(3, \frac{1}{2}\right)$.

Extra question 1a – 6 points

$f'(x) = \dfrac{1}{2\sqrt{9-2x}} \cdot -2 = -\dfrac{1}{\sqrt{9-2x}} \Rightarrow f'(0) = -\frac{1}{3}$

$f(0) = 3$, an equation of the tangent line is therefore $y = -\frac{1}{3}x + 3$

Intersection with the *x*-axis: $x = 9$

Area triangle: $\frac{1}{2} \cdot 3 \cdot 9 = 13\frac{1}{2}$

Extra question 1b – 6 points

$f(x) = 0 \Leftrightarrow 9 - 2x = 0 \Leftrightarrow x = 4\frac{1}{2}$; so we have to compute $\int_0^{4{,}5} f(x)\ \mathrm{d}x$

The antiderivative has the form $F(x) = a(9-2x)^{\frac{3}{2}}$ with $a = \frac{1}{3/2} \cdot \frac{1}{-2} = -\frac{1}{3}$

$F(0) = -\frac{1}{3} \cdot 9^{\frac{3}{2}} = -9$; $F\left(4\frac{1}{2}\right) = 0$. The area is therefore $0 - (-9) = 9$

Extra question 1c – 5 points

$f(x) = g(x) \Leftrightarrow \sqrt{9-2x} = 2x + 11 \Rightarrow 9 - 2x = (2x+11)^2 \Leftrightarrow 9 - 2x = 4x^2 + 44x + 121$

$$\Leftrightarrow 4x^2 + 46x + 112 = 0 \Leftrightarrow x = \frac{-46 + \sqrt{324}}{8} = \frac{-46+18}{8} = -3\frac{1}{2} \vee x = \frac{-46-18}{8} = -8$$

Only $x = -3\frac{1}{2}$ suffices.

Extra question 2a – 7 points

$f(x) = g(x) \Leftrightarrow \cos(2x) = \sin\left(2x - \frac{1}{6}\pi\right) \Leftrightarrow \cos(2x) = \cos\left(2x - \frac{1}{6}\pi - \frac{1}{2}\pi\right) \Leftrightarrow \cos(2x) = \cos\left(2x - \frac{2}{3}\pi\right)$

The equation can also be transformed into $\sin\left(2x + \frac{1}{2}\pi\right) = \sin\left(2x - \frac{1}{6}\pi\right)$

This yields $2x = -2x + \frac{2}{3}\pi + k \cdot 2\pi \Leftrightarrow 4x = \frac{2}{3}\pi + k \cdot 2\pi \Leftrightarrow x = \frac{1}{6}\pi + k \cdot \frac{1}{2}\pi$

Solutions on the interval $[-\pi, \pi]$: $x = \frac{1}{6}\pi$; $x = \frac{2}{3}\pi$; $x = -\frac{1}{3}\pi$; $x = -\frac{5}{6}\pi$

Extra question 2b – 7 points

Direct differentiation of $f(x) = \sin^2(x) + \cos(2x)$ yields

$f'(x) = 2\sin(x)\cos(x) - 2\sin(2x) = \sin(2x) - 2\sin(2x) = -\sin(2x)$

or $f'(x) = 2\sin(x)\cos(x) - 2\sin(2x) = 2\sin(x)\cos(x) - 2 \cdot 2\sin(x)\cos(x) = -2\sin(x)\cos(x)$

Alternative approach:

$f(x) = \sin^2(x) + \cos(2x) = \sin^2(x) + \cos^2(x) - \sin^2(x) = \cos^2(x)$ also yields $f'(x) = -2\sin(x)\cos(x)$

In all cases we get $f'(x) = 0 \Leftrightarrow x = 0 + k \cdot \frac{1}{2}\pi$

Maxima for $x = 0$ $(+k \cdot \pi)$: $f(0) = 1$

Minima for $x = \frac{1}{2}\pi$ $(+k \cdot \pi)$: $f\left(\frac{1}{2}\pi\right) = 0$

Extra question 2c – 4 points

On one side we have:

$$h(x) = f(x) + g(x) + 2\cos^2(x) - 2 = \sin^2(x) + \cos(2x) + \sin^2(x) + \sin\left(2x - \frac{1}{6}\pi\right) + 2\cos^2(x) - 2$$

$$= \cos(2x) + \sin\left(2x - \frac{1}{6}\pi\right) + 2(\sin^2(x) + \cos^2(x)) - 2 = \cos(2x) + \sin\left(2x - \frac{1}{6}\pi\right)$$

$$= \cos(2x) + \sin(2x)\cos(\tfrac{1}{6}\pi) - \cos(2x)\sin\left(\frac{1}{6}\pi\right) = \frac{1}{2}\cos(2x) - \frac{1}{2}\sqrt{3} \cdot \sin(2x)$$

On the other side we have:

$$\sin\left(2x + \frac{1}{6}\pi\right) = \sin(2x)\cos\left(\frac{1}{6}\pi\right) + \cos(2x) \cdot \sin\left(\frac{1}{6}\pi\right) = -\frac{1}{2}\sqrt{3} \cdot \sin(2x) + \frac{1}{2}\cos(2x)$$

Extra question 2d – 4 points

$$\int_0^{\frac{\pi}{2}} \sin\left(2x + \frac{1}{6}\pi\right)\,\mathrm{d}x = \left[-\frac{1}{2}\cos\left(2x + \frac{1}{6}\pi\right)\right]_0^{\frac{\pi}{2}} = -\frac{1}{2}\cos\left(1\tfrac{1}{6}\pi\right) + \frac{1}{2}\cos\left(\frac{1}{6}\pi\right) = -\frac{1}{2} \cdot -\frac{1}{2}\sqrt{3} + \frac{1}{2} \cdot \frac{1}{2}\sqrt{3} = \frac{1}{2}\sqrt{3}$$