Elaborations Example Exam 1 Wiskunde B 2018

Question 1a – 4 points

$x(t) = 0 \Leftrightarrow 1 - t^2 = 0 \Leftrightarrow t = \pm 1$

$t = -1$ yields $y(t) = 0$; $t = 1$ yields $y(t) = (1+1)^2 = 4$ so in point *A* we have $t = 1$

$x'(t) = -2t$; $y'(t) = 2(t+1)$, so $x'(1) = -2$ and $y'(1) = 4$

This yields $v = \sqrt{(-2)^2 + 4^2} = \sqrt{20}$

Question 1b – 4 points

$$\big(x(t) + y(t)\big)^2 = (1 - t^2 + (1+t)^2)^2 = (1 - t^2 + 1 + 2t + t^2)^2$$
$$= (2 + 2t)^2 = \big(2(1+t)\big)^2 = 2^2 \cdot (1+t)^2 = 4y(t)$$

Question 1c – 4 points

$\vec{v} = \begin{pmatrix} x'(1) \\ y'(1) \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$. This is the normal vector of line ℓ

So the equation has the form $-2x + 4y = c$.

Substitution of $x_A = 0$ and $y_A = 4$ yields $c = 16$, so the equation is $-2x + 4y = 16 \Leftrightarrow y = \frac{1}{2}x + 4$

Alternative:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y'(1)}{x'(1)} = \frac{4}{-2} = -2$$

So the slope of line ℓ fulfils $-2 \cdot a = -1 \Leftrightarrow a = \frac{1}{2}$

The equation of the line through $A(0,4)$ with slope $\frac{1}{2}$ is $y = \frac{1}{2}x + 4$

Question 2a – 3 points

$F'(x) = 1 \cdot \mathrm{e}^{-x} + (x+1) \cdot -\mathrm{e}^{-x} = \mathrm{e}^{-x} - x \cdot \mathrm{e}^{-x} - \mathrm{e}^{-x} = -x \cdot \mathrm{e}^{-x} = f_0(x)$

Question 2b – 6 points

$$\int_0^1 f_1(x)\mathrm{d}x = \int_0^1 (1-x)\mathrm{e}^{-x}\mathrm{d}x = \int_0^1 \mathrm{e}^{-x} - x\mathrm{e}^{-x}\mathrm{d}x = \int_0^1 \mathrm{e}^{-x} + f_0(x)\mathrm{d}x$$

$$= [-\mathrm{e}^{-x} + (1+x)\mathrm{e}^{-x}]_0^1 = [x\mathrm{e}^{-x}]_0^1 = \mathrm{e}^{-1} - 0 = \frac{1}{\mathrm{e}}$$

Question 2c – 5 points

$$\int_0^1 f_p(x) - f_{-p}(x)\,\mathrm{d}x = \int_0^1 (p-x)\mathrm{e}^{-x} - (-p-x)\mathrm{e}^{-x}\mathrm{d}x = \int_0^1 2p\mathrm{e}^{-x}\mathrm{d}x$$

$$= [-2p\mathrm{e}^{-x}]_0^1 = -2p\mathrm{e}^{-1} + 2p = 2p(1-\mathrm{e}^{-1})$$

$2p(1-\mathrm{e}^{-1}) = 2 \Leftrightarrow p = \dfrac{1}{1-\mathrm{e}^{-1}} = \dfrac{\mathrm{e}}{\mathrm{e}-1}$

Question 2d – 6 points

$f_p'(x) = -1 \cdot \mathrm{e}^{-x} + (p-x) \cdot -\mathrm{e}^{-x} = -\mathrm{e}^{-x} - p\mathrm{e}^{-x} + x\mathrm{e}^{-x}$

$f_p''(x) = \mathrm{e}^{-x} + p\mathrm{e}^{-x} + 1 \cdot \mathrm{e}^{-x} + x \cdot -\mathrm{e}^{-x} = (2+p-x)\mathrm{e}^{-x}$

$f_p''(x) = 0 \Leftrightarrow 2 + p - x = 0 \Leftrightarrow p = x - 2$

This yields $y = f_{x-2}(x) = \big((x-2) - x\big) \cdot \mathrm{e}^{-x} = -2\mathrm{e}^{-x}$

Question 3a – 6 points

The equation of circle *c* is $(x-14)^2+(y-8)^2=10^2$

$y=0$ yields $(x-14)^2+8^2=10^2 \Leftrightarrow x^2-28x+196+64=100 \Leftrightarrow x^2-28x+160=0$

This yields $(x-8)(x-20)=0$ so $x_A=8$ en $x_B=20$

Of course, the equation can also be solved with the quadratic formula.
De coordinates of A and B can also be found with Pythagoras in triangles APC and BPC, where $P(14{,}0)$ *is the projection of M on the x-axis.*

The radius of c^* is *r*; $P(14{,}0)$ is the projection of *M* on the *x*-axis.
The centre *Q* of circle c^* is on the line segment bisector of *A* and *B*, that is the line $x=14$.

$|MQ|=r$ yields $|PQ|=8-r$.

Pythagoras in triangle APQ yields: $|AP|^2+|PQ|^2=|AQ|^2$

This yields: $(14-8)^2+(8-r)^2=r^2 \Leftrightarrow 36+64-16r+r^2=r^2 \Leftrightarrow 16r=100 \Leftrightarrow r=6{,}25$

Alternative 1:

Computation of the coordinates of *A* and *B* as above.

The line segment bisector of $A(8{,}0)$ and $B(20{,}0)$ is the vertical line $x=14$

The straight line through $A(8{,}0)$ and $M(14{,}8)$ has slope $\frac{4}{3}$

Therefore, the line segment bisector of *A* and *M* is the line through $(11{,}4)$ with slope $-\frac{3}{4}$

The equation of this line is $y-4=-\frac{3}{4}(x-11) \Leftrightarrow y=-\frac{3}{4}x+12\frac{1}{4}$

Q, the centre of c^* is the intersection of these line segment bisectors.

$y_Q=-\frac{3}{4}x_Q+12\frac{1}{4} \wedge x_Q=14$ yields $y_Q=1\frac{3}{4}$

$r=|MQ|=8-1\frac{3}{4}=6\frac{1}{4}$

Alternative 2:

Computation of the coordinates of *A* and *B* as above.

Substitution of the coordinates of $A(8{,}0)$, $B(20{,}0)$ and $M(14{,}8)$ into $(x-a)^2+(y-b)^2=r^2$ yields three equations in three unknowns from which *r* can be solved.

Question 3b – 4 points

The radius of *d* is *r*; $P(14{,}0)$ is the projection of *M* on the *x*-axis.
This yields $NP=14-r$; $PM=8$ and $NM=r+10$

Pythagoras then yields

$(14-r)^2+8^2=(r+10)^2 \Leftrightarrow 196-28r+r^2+64=r^2+20r+100 \Leftrightarrow 48r=160 \Leftrightarrow r=\frac{10}{3}=3\frac{1}{3}$

Question 4a – 5 points

$f(x) = g(x) \Leftrightarrow 3\ln(x) = \ln^3(x) \Leftrightarrow 3\ln(x) - \ln^3(x) = 0 \Leftrightarrow \ln(x)\,(3 - \ln^2(x)) = 0$

This yields $\ln(x) = 0 \vee \ln^2(x) = 3$

$\ln(x) = 0 \Leftrightarrow x = 1 \wedge y = f(1) = 0$, so *B* is point (1,0)

$\ln^2(x) = 3 \Leftrightarrow \ln(x) = \pm\sqrt{3} \Leftrightarrow x = \mathrm{e}^{\sqrt{3}} \vee x = \mathrm{e}^{-\sqrt{3}}$

$x = \mathrm{e}^{-\sqrt{3}}$ geeft $y = -3\sqrt{3}$ so *A* is point $\left(\mathrm{e}^{-\sqrt{3}}, -3\sqrt{3}\right)$

$x = \mathrm{e}^{\sqrt{3}}$ geeft $y = 3\sqrt{3}$ dus *C* is point $\left(\mathrm{e}^{\sqrt{3}}, 3\sqrt{3}\right)$

Question 4b – 6 points

In the graph we can see that the distance between these points on this interval is given by $A(p) = f(p) - g(p) = 3\ln(p) - (\ln(p))^3$ *and that this function indeed had a maximum.*

$A'(p) = \frac{3}{p} - \frac{3}{p}(\ln(p))^2$

$A'(p) = 0 \Leftrightarrow (\ln(p))^2 = 1$

This yields $\ln(p) = 1 \Leftrightarrow p = \mathrm{e}$

The solution $\ln(p) = -1 \Leftrightarrow p = \mathrm{e}^{-1}$ *is not in the interval.*

Therefore, the maximal distance is $A(\mathrm{e}) = 3 - 1 = 2$

Question 4c – 6 points

$y = f(x) = 3\ln(x) \Leftrightarrow \ln(x) = \frac{1}{3}y \Leftrightarrow x = \mathrm{e}^{\frac{1}{3}y}$

Tis yields $\pi \cdot \int_0^1 x^2\,\mathrm{d}y = \pi \cdot \int_0^1 \left(\mathrm{e}^{\frac{1}{3}y}\right)^2 \mathrm{d}y = \pi \cdot \int_0^1 \mathrm{e}^{\frac{2}{3}y}\,\mathrm{d}y = \pi \cdot \left[\frac{3}{2}\mathrm{e}^{\frac{2}{3}y}\right]_0^1 = \frac{3}{2}\pi \cdot \left(\mathrm{e}^{\frac{2}{3}} - 1\right)$

Question 5a – 4 points

In the perforation, both the numerator and the denominator of $f(x)$ are 0.

$(x-1)(x^2+x+1)=0 \Leftrightarrow x-1=0 \Leftrightarrow x=1$ (The discriminant of the other factor is negative!)

$x=1$ yields $x^2-1=0$.

Since $x^2-1=(x-1)(x+1)$, for $x \neq 1$ we have $f(x)=\frac{x^2+x+1}{x+1}$

This yields $\lim_{x\to 1} f(x)=\frac{1+1+1}{1+1}=\frac{3}{2}$

The coordinates of the perforation therefore are $x=1$ and $y=\frac{3}{2}$

Question 5b – 2 points

Vertical asymptote: $x=-1$

since for $x=-1$ the denominator is 0 and the numerator is 1

Question 5c – 4 points

For $x \neq 1$ we have $f(x)=\frac{x^2+x+1}{x+1}$

$$\frac{x^2+x+1}{x+1}=\frac{x^2+x}{x+1}+\frac{1}{x+1}=\frac{x(x+1)}{x+1}+\frac{1}{x+1}=x+\frac{1}{x+1}$$

so $\lim_{x\to\pm\infty} f(x)-x=\lim_{x\to\pm\infty}\frac{1}{x+1}=0$

The oblique asymptote is therefore $y=x$

Alternative:

$$\frac{(x-1)(x^2+x+1)}{x^2-1}=\frac{x^3-1}{x^2-1}=\frac{x(x^2-1)}{x^2-1}+\frac{x-1}{x^2-1}=x+\frac{x-1}{x^2-1}=x+\frac{\frac{1}{x}-\frac{1}{x^2}}{1-\frac{1}{x^2}}$$

so $\lim_{x\to\pm\infty} f(x)-x=\lim_{x\to\pm\infty}\frac{\frac{1}{x}-\frac{1}{x^2}}{1-\frac{1}{x^2}}=\frac{0-0}{1-0}=0$

with the same conclusion as above.

Question 5d – 5 points

$$f(\quad)=\frac{x^2+x+1}{x+1}\Rightarrow f'(x)=\frac{(2x+1)(x+1)-(x^2+x+1)\cdot 1}{(x+1)^2}=\frac{2x^2+3x+1-x^2-x-1}{(x+1)^2}$$

$$=\frac{x^2+2x}{(x+1)^2}$$

$f'(x)=0 \Leftrightarrow x^2+2x=0 \Leftrightarrow x(x+2)=0 \Leftrightarrow x=0 \vee x=-2$

For $x=0$, f has a minimum.

Therefore, the coordinates of the vertex where f has a maximum are $x=-2$ and $y=f(-2)=-3$

Question 6a – 4 points

In the vertical asymptotes we have $\sin\left(2x-\frac{2}{3}\pi\right)=0$

This yields $2x=\frac{2}{3}\pi+k\cdot\pi \Leftrightarrow x=\frac{1}{3}\pi+k\cdot\frac{1}{2}\pi$

In the figure are $x=-\frac{1}{6}\pi$; $x=\frac{1}{3}\pi$ and $x=\frac{5}{6}\pi$

Question 6b – 6 points

$f'(x)=-\frac{1}{\left(\sin\left(2x-\frac{2}{3}\pi\right)\right)^2}\cdot 2\cos\left(2x-\frac{2}{3}\pi\right)$

$f'(x)=0 \Leftrightarrow \cos\left(2x-\frac{2}{3}\pi\right)=0$

This yields $2x-\frac{2}{3}\pi=\frac{1}{2}\pi+k\cdot\pi \Leftrightarrow x=\frac{7}{12}\pi+k\cdot\frac{1}{2}\pi$

$x=\frac{7}{12}\pi+k\cdot\pi$ yields $f(x)=\frac{1}{\sin\frac{1}{2}\pi}=1$ and $g(x)=\frac{\sin\left(\frac{1}{4}\pi\right)}{\cos\left(\frac{1}{4}\pi\right)}=1$

$x=\frac{1}{12}\pi+k\cdot\pi$ yields $f(x)=\frac{1}{\sin-\frac{1}{2}\pi}=-1$ and $g(x)=\frac{\sin\left(\frac{3}{4}\pi\right)}{\cos\left(\frac{3}{4}\pi\right)}=-1$

Alternative:

f has a minimum when $\frac{1}{f(x)}$ has a maximum and a maximum when $\frac{1}{f(x)}$ has a minimum.

Therefore, *f* has a minimum when $\sin\left(2x-\frac{2}{3}\pi\right)=1$

This is when $2x-\frac{2}{3}\pi=\frac{1}{2}\pi+k\cdot 2\pi \Leftrightarrow 2x=\frac{7}{6}\pi+k\cdot 2\pi \Leftrightarrow x=\frac{7}{12}\pi+k\cdot\pi$

And *f* has a maximum when $\sin\left(2x-\frac{2}{3}\pi\right)=-1$

This is when $2x-\frac{2}{3}\pi=-\frac{1}{2}\pi+k\cdot 2\pi \Leftrightarrow 2x=\frac{1}{6}\pi+k\cdot 2\pi \Leftrightarrow x=\frac{1}{12}\pi+k\cdot\pi$

In the minima we have $f(x)=\frac{1}{1}=1$ and $g(x)=\frac{\sin\left(\frac{1}{4}\pi\right)}{\cos\left(\frac{1}{4}\pi\right)}=1$

In the maxima we have $f(x)=\frac{1}{-1}=-1$ and $g(x)=\frac{\sin\left(\frac{3}{4}\pi\right)}{\cos\left(\frac{3}{4}\pi\right)}=-1$

Question 6c – 5 points

$f(x)=h(x) \Leftrightarrow \frac{1}{\sin\left(2x-\frac{2}{3}\pi\right)}=4\cos\left(2x-\frac{2}{3}\pi\right) \Leftrightarrow 4\sin\left(2x-\frac{2}{3}\pi\right)\cos\left(2x-\frac{2}{3}\pi\right)=1$

$\sin(2A)=2\sin(A)\cos(A)$ then yields $\sin\left(4x-\frac{4}{3}\pi\right)=\frac{1}{2}$

This yields $4x-\frac{4}{3}\pi=\frac{1}{6}\pi+k\cdot 2\pi$ or $4x-\frac{4}{3}\pi=\frac{5}{6}\pi+k\cdot 2\pi$

$4x-\frac{4}{3}\pi=\frac{1}{6}\pi+k\cdot 2\pi \Leftrightarrow 4x=\frac{3}{2}\pi+k\cdot 2\pi \Leftrightarrow x=\frac{3}{8}\pi+k\cdot\frac{1}{2}\pi$

$4x-\frac{4}{3}\pi=\frac{5}{6}\pi+k\cdot 2\pi \Leftrightarrow 4x=\frac{13}{6}\pi+k\cdot 2\pi \Leftrightarrow x=\frac{13}{24}\pi+k\cdot\frac{1}{2}\pi$

Extra question, item a – 4 points

$f(x) = 7 \Leftrightarrow x - 3 + \dfrac{4}{x+2} = 7 \Leftrightarrow (x-3)(x+2) + 4 = 7(x+2)$

$$\Leftrightarrow x^2 - x - 6 + 4 = 7x + 14 \Leftrightarrow x^2 - 8x - 16 = 0 \Leftrightarrow x = \frac{8 \pm \sqrt{128}}{2} \quad \left(= 4 \pm 4\sqrt{2}\right)$$

Extra question, item b – 5 points

With discriminant:

$f(x) = p \Leftrightarrow x - 3 + \dfrac{4}{x+2} = p \Leftrightarrow (x-3)(x+2) + 4 = p(x+2)$

$$\Leftrightarrow x^2 - x - 6 + 4 = px + 2p \Leftrightarrow x^2 + (-1-p)x + (-2-2p) = 0$$

There are no common points when de discriminant of this equation is negative

$D = (-1-p)^2 - 4(-2-2p) = 1 + 2p + p^2 + 8 + 8p = p^2 + 10p + 9$

$D = 0 \Leftrightarrow p = -1 \vee p = -9;\; D < 0 \Leftrightarrow -9 < p < -1$

Since the graph of $D(p) = p^2 + 10p + 9$ *is an upward opening parabola.*

With derivative:

$f'(x) = 1 - \dfrac{4}{(x+2)^2}$

$f'(x) = 0 \Leftrightarrow \dfrac{4}{(x+2)^2} = 1 \Leftrightarrow (x+2)^2 = 4 \Leftrightarrow x + 2 = \pm 2 \Leftrightarrow x = 0 \vee x = -4$

$f(0) = -1;\; f(-4) = -9$

In the figure we can see that there are no common points for $-9 < p < -1$

Extra question, item c – 6 points

$f'(x) = 1 - \dfrac{4}{(x+2)^2}$

$f'(x) = -3 \Leftrightarrow 1 - \dfrac{4}{(x+2)^2} = -3 \Leftrightarrow \dfrac{-4}{(x+2)^2} = -4 \Leftrightarrow (x+2)^2 = 1 \Leftrightarrow x + 2 = \pm 1 \Leftrightarrow x = -1 \vee x = -3$

ℓ is the tangent line $x = -1$, so *m* is the tangent line for $x = -3$.

$f(-3) = -10$, so tangent line *m* has equation $y + 10 = -3(x+3)$

Or:

$y = ax + b$ *with* $y = -10$, $a = -3$ *and* $x = -3$ *yields* $-10 = 9 + b \Leftrightarrow b = -19$, *so* $y = -3x - 19$

Extra question, item d – 7 points

$$\int_{-1}^{2} -f(x)\,\mathrm{d}x = \int_{-1}^{2} -x + 3 - \frac{4}{x+2}\,\mathrm{d}x = \left[-\tfrac{1}{2}x^2 + 3x - 4\ln(x+2)\right]_{-1}^{2}$$

$$= -2 + 6 - 4\ln(4) + \tfrac{1}{2} + 3 + 0 = 7\tfrac{1}{2} - 4\ln(4)$$

Extra item for question 1

$\vec{v} = \begin{pmatrix} x'(1) \\ y'(1) \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$ is perpendicular to ℓ

The direction vector ℓ is therefore $\begin{pmatrix} 4 \\ -(-2) \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

Since line ℓ passes through point $A(0,4)$, this yields the vector representation $\vec{x} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

Extra item for question 6

$$g'(x) = \frac{-\cos\left(\frac{5}{6}\pi - x\right) \cdot \cos\left(\frac{5}{6}\pi - x\right) - \sin\left(\frac{5}{6}\pi - x\right) \cdot \left(-\left(-\sin\left(\frac{5}{6}\pi - x\right)\right)\right)}{\left(\cos\left(\frac{5}{6}\pi - x\right)\right)^2} = -\frac{1}{\left(\cos\left(\frac{5}{6}\pi - x\right)\right)^2}$$

$$g'\left(\tfrac{2}{3}\pi\right) = -\frac{1}{\left(\cos\left(\frac{1}{6}\pi\right)\right)^2} = -\frac{1}{\frac{3}{4}} = -\tfrac{4}{3}$$

$$g\left(\tfrac{2}{3}\pi\right) = \frac{\sin\left(\frac{1}{6}\pi\right)}{\cos\left(\frac{1}{6}\pi\right)} = \tfrac{1}{3}\sqrt{3}$$

The tangent line is found by $y - \frac{1}{3}\sqrt{3} = -\frac{4}{3}\left(x - \frac{2}{3}\pi\right)$

or by substitution of $y = \frac{1}{3}\sqrt{3}$, $x = \frac{2}{3}\pi$ and $a = -\frac{4}{3}$ into $y = ax + b$

This yields $y = -\frac{4}{3}x + \frac{8}{9}\pi - \frac{1}{3}\sqrt{3}$