CENTRALE COMMISSIE VOORTENTAMEN WISKUNDE

Entrance Exam Wiskunde A

Date:	17 July 2021
Time:	140 minutes (2 hours and 20 minutes)
Questions:	6

Please read the instructions below carefully before answering the questions. Failing to comply with these instructions may result in deduction of points.

Make sure your name is clearly written on every answer sheet.

Take a new answer sheet for every question.

Show all your calculations clearly. Illegible answers and answers without a calculation or an explanation of the use of your calculator are invalid (*see also question 1)*.

Write your answers in ink. Do not use a pencil, except when drawing graphs. Do not use correction fluid.

You can use a basic scientific calculator. **Other equipment, like a graphing calculator, a calculator with the option of computing integrals, a formula chart, BINAS or a book with tables, is NOT permitted.**

On the last two pages of this exam you will find a list of formulas.

You can use a dictionary if it is approved by the invigilator.

Please **switch off your mobile telephone** and put it in your bag.

Because the time for this exam has been reduced to 140 minutes, the number of items per question has been reduced. Therefore, the total number of points that can be scored is reduced to 72.

Points that can be scored for each item:

Question	1	2	3	4	5	6
a	5	5	3	4	5	4
b	5	5	5	2	2	4
c	5		1	4	4	
d			4	3		
e			2			
Total	15	10	15	13	11	8

Grade $= \dfrac{\text{total points scored}}{8} + 1$

You will pass the exam if your grade is at least 5.5 .

Question 1 – Algebraic computations

Take a new answer sheet for every question!

When you are asked to perform a computation ***algebraically****, your computation should be fully worked out on paper. Reading function values from a table (including tables produced by a calculator) is not allowed in algebraic calculations. You can use a calculator for simple calculations and for approximations of numbers like* $\sqrt{2}$ *and* $\log(3)$.
Unless stated otherwise, all computations in this exam have to be performed algebraically.

The function f is given by $f(x) = x^4 + 2x^3 - 2x^2 + 4$.

5pt a Compute algebraically the *x*-coordinates of the points on the graph of f where the tangent line is horizontal.

The functions g and h are given by $g(x) = 3 \cdot 2^x$ and $h(x) = 12^x$.

5pt b Compute algebraically the *x*-coordinate of the intersection of the graphs of g and h. Give an approximation of your answer rounded to two digits behind the decimal point.

The function k is given by $k(x) = \sqrt{3x^2 - 4}$.

5pt c Compute algebraically the value(s) of x for which $k'(x) = 3$.

Question 2 – The spread of an influenza virus

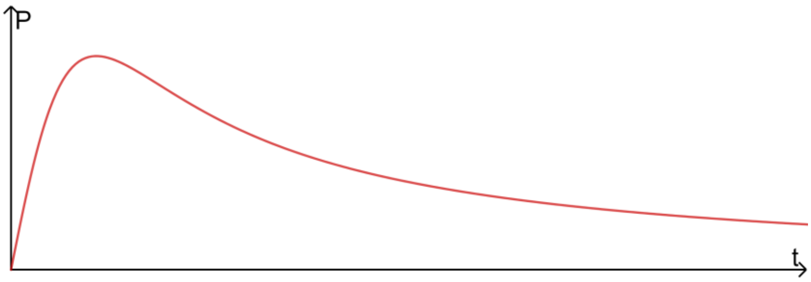
Take a new answer sheet for every question!

In normal winters, a new variant of the influenza virus often develops. In a given year, the percentage of the population of country C that has to stay at home sick because they are infected with the new variant of the influenza virus from that year is modelled with the formula

$$P = \frac{100t}{400 + t^2}$$

with t in days.

In the figure below, the graph of P that fits this formula is shown.



5pt **a** Compute algebraically the maximal percentage of the population of country C that has to stay at home sick because they are infected with the new variant of the influenza virus.

5pt **b** Compute algebraically the number of days on which more than 2% of the population of country C has to stay at home sick because they are infected with the new variant of the influenza virus.

Question 3 – Two dice

Take a new answer sheet for every question!

In a game, two dice are rolled. One of these dice is a regular die, with six faces on which the numbers 1 to 6 are written. The other is a special die: it also has six faces, but one of the faces has the number 1, two faces have the number 2, and the other three faces have the number 3. At each round of the game, both dice are rolled and then the sum of the numbers rolled is counted.

In questions a and b, we assume that both dice are "fair", that is, each face has an equal chance of being rolled.

3pt a Compute the probability that the sum of the numbers that are rolled in a game is equal to 7.

Two children, Astrid and Bartje, play the following game. In each round, both dice are rolled once; if the sum of the numbers on the two dice is even, Astrid gets a point, and if that sum is odd, Bartje gets a point. Whoever collects three points first wins.

5pt b Compute the probability that this game will be completed in exactly four rounds.

Over time, it appears that the sum of the figures is less often equal to 7 than might be expected on the basis of the outcome of question a. Astrid suspects that this is because the special die has a greater chance of rolling the number 1 than if it were a fair die. To test this, she rolls the special die 36 times. In these 36 rolls, the number 1 is rolled 10 times.

1pt c State the null hypothesis and the alternative hypothesis for this test procedure.

4pt d Compute the probability that if the special die is fair, the number 1 will be rolled in 10 out of 36 times.

2pt e Can you draw a conclusion for this test procedure based on your answer of item d? If so, state and motivate this conclusion. If not, explain why not.

Question 4 – White-tailed eagles in the Netherlands

Take a new answer sheet for every question!

White-tailed eagles are nicknamed "flying doors" because of their huge and massive wings that can span almost 2.5 meters. White-tailed eagles were not found in the Netherlands for a long time, until the first pair of white-tailed eagles settled in the Oostvaardersplassen (a new nature reserve in the province of Flevoland) in 2006.

In eleven years, from 2009 to 2020, the number of breeding pairs has increased approximately exponentially from 2 to 15. We therefore use a model with an exponential relationship between A, the number of breeding pairs, and t, the time in years, with $A = 2$ at $t = 0$ and $A = 15$ at $t = 11$.

4pt **a** Compute algebraically by what percentage per year the number of breeding pairs has increased according to this model from 2009 to 2020.

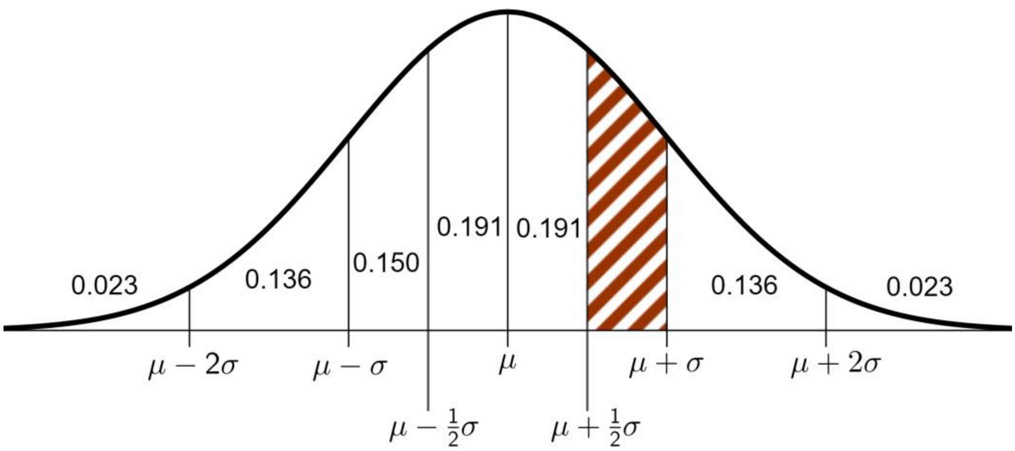
After a young white-tailed eagle has hatched (this is called the day of birth), it takes 70 to 90 days for the young to leave the nest. This time period is called the flight time and it is normally distributed with mean $\mu = 80$ days and standard deviation $\sigma = \frac{10}{3}$ days.

2pt **b** Use the figure below to explain that there are virtually no young white-tailed eagles that have a flight time of less than 70 or more than 90 days.

4pt **c** Use the figure below to compute the probability that a young white-tailed eagle has a flight time of less than 81 days and 16 hours.

In 2020, 20 young white-tailed eagles fledged in the Netherlands. The total flight time of these 20 young is a normally distributed random variable *T*.

3pt **d** Compute μ_T and σ_T.

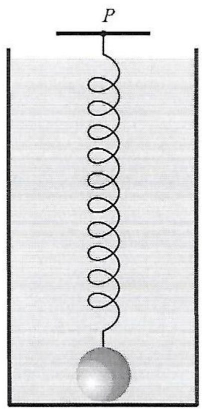


A normal probability distribution X

The area of the shaded region represents $P\left(\mu + \frac{1}{2}\sigma < X < \mu + \sigma\right) = 0.150$

Question 5 – Sphere in reservoir with a viscous liquid

Take a new answer sheet for every question!



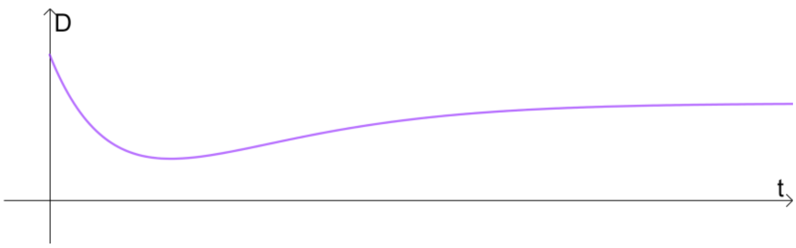
The figure on the right shows a reservoir filled with a viscous liquid. A sphere is located in the reservoir, attached to the bottom. The sphere is connected to a point *P* by an elongated spring.

When the sphere is released from the bottom of the reservoir, the sphere starts a vertical upward movement. Research has shown that t seconds after the time of release, the distance D from the centre of the sphere to point *P* is given by the formula

$$D = 10 + (5 - t)\mathrm{e}^{-0.05t}$$

In this formula, D is in centimeters.

The figure below shows the graph of D that fits this formula.



As you can see in the graph, D is minimal at a certain moment. The sphere is then at its highest point. After that moment, it slowly sinks back into the liquid.

5pt **a** Using the derivative function, show that the sphere reaches the minimal distance to point *P* at $t = 25$.

In the long run, the sphere comes to a standstill in the liquid.

2pt **b** Compute the distance between the centre of the sphere and *P* that is reached in the long run.

The height of the reservoir is equal to the distance from *P* to the bottom of the reservoir. The volume of the sphere is 33.51 cm^3. The relationship between the volume V and the radius r of the sphere is given by $V = \frac{4}{3}\pi r^3$.

4pt **c** Compute algebraically the height of the reservoir in cm.
Hint: first compute the radius r of the sphere.

Question 6 – Sphere in vacuum vessel

Take a new answer sheet for every question!

In this question, the sphere from question 5 is not suspended in a reservoir with a viscous liquid, but in a vessel that has been evacuated. The movement of the sphere in this vacuum vessel is not slowed down, so the sphere continues to move up and down after it has been released from the bottom. The distance between the centre of the sphere and the attachment point *P* of the spring is now given by a formula of the form

$$D = a + b\sin\left(ct + \tfrac{1}{2}\pi\right)$$

In this formula, D is again in cm and t is again in seconds.

At $t = 0$ the sphere is released from the bottom. The distance between *P* and the centre of the sphere is equal to 25 cm. This is the maximum value of D.
At $t = 12$ the sphere reaches its highest point for the first time. The distance between *P* and the centre of the sphere is therefore minimal for the first time. This minimal distance is 12 cm.

4pt a Compute algebraically the values of a, b and c in the formula for D.

At $t = 8$, the distance between the centre of the sphere and point *P* is 15.25 cm.

4pt b Compute algebraically the first three times after $t = 8$ at which this distance is 15.25 cm.

End of the exam.

When you have finished the exam, check whether your ***name*** *and the* ***question number*** *are on every answer sheet.*

Place the answer sheets in the correct order in the plastic folder and place the sheet with your data in the front in this folder.

What should ***not*** *be in the folder:*
- empty sheets, please leave them on your table;
- sheets with only your name on it, please take them with you;
- scrap paper;
- these questions.

This is the only way we can ensure a smooth correction of your exam work.

Remain seated until one of the invigilators collects your folder (or calls you).

Formula list wiskunde A

Quadratic equations

The solutions of the equation $ax^2 + bx + c = 0$ with $a \neq 0$ and $b^2 - 4ac \geq 0$ are

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Differentiation

Rule	**function**	**derivative function**
Sum rule	$s(x) = f(x) + g(x)$	$s'(x) = f'(x) + g'(x)$
Product rule	$p(x) = f(x) \cdot g(x)$	$p'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
Quotient rule	$q(x) = \frac{f(x)}{g(x)}$	$q'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{\left(g(x)\right)^2}$
Chain rule	$k(x) = f(g(x))$	$k'(x) = f'\left(g(x)\right) \cdot g'(x)$ or $\frac{dk}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$

Logarithms

Rule	**conditions**
${}^{g}\log a + {}^{g}\log b = {}^{g}\log ab$	$g > 0, \quad g \neq 1, \quad a > 0, \quad b > 0$
${}^{g}\log a - {}^{g}\log b = {}^{g}\log \frac{a}{b}$	$g > 0, \quad g \neq 1, \quad a > 0, \quad b > 0$
${}^{g}\log a^p = p \cdot {}^{g}\log a$	$g > 0, \quad g \neq 1, \quad a > 0$
${}^{g}\log a = \frac{{}^{p}\log a}{{}^{p}\log g}$	$g > 0, \quad g \neq 1, \quad a > 0, \quad p > 0, \quad p \neq 1$

Arithmetic and geometric sequences

Arithmetic sequence:	$Sum = \frac{1}{2} \cdot number\ of\ terms \cdot (u_e + u_l)$
Geometric sequence:	$Sum = \frac{u_{l+1} - u_e}{r - 1} \quad (r \neq 1)$
In both formulas:	*e* = number first term of the sum; *l* = number last term of the sum

More formulas on the next page.

Formula list wiskunde A (continued)

Probability

If *X* and *Y* are any random variables, then: $E(X+Y) = E(X) + E(Y)$ If furthermore *X* and *Y* are independent, then: $\sigma(X+Y) = \sqrt{\sigma^2(X) + \sigma^2(Y)}$
$\sqrt{n}$-law: For *n* independent repetitions of the same experiment where the result of each experiment is a random variable *X*, the sum of the results is a random variable *S* and the mean of the results is a random variable $\overline{X}$. $E(S) = n \cdot E(X)$ $\sigma(S) = \sqrt{n} \cdot \sigma(X)$ $E(\overline{X}) = E(X)$ $\sigma(\overline{X}) = \dfrac{\sigma(X)}{\sqrt{n}}$

Binomial Distribution

If *X* has a binomial distribution with parameters *n* (number of experiments) and *p* (probability of success at each experiment), then

$$P(X = k) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} \quad \text{with} \quad k = 0, 1, 2, \dots, n$$

Expected value: $E(X) = np$ Standard deviation: $\sigma(X) = \sqrt{n \cdot p \cdot (1-p)}$

n and *p* are the parameters of the binomial distribution

Normal Distribution

If *X* is a normally distributed random variable with mean μ and standard deviation σ, then

$Z = \frac{X-\mu}{\sigma}$ has a standard normal distribution and $P(X < g) = P\left(Z < \frac{g-\mu}{\sigma}\right)$

μ and σ are the parameters of the normal distribution.

Hypothesis testing

In a testing procedure where the test statistic *T* is normally distributed with mean μ_T standard deviation σ_T the boundaries of the rejection region (the critical region) are:

α	left sided	right sided	two sided
0.05	$g = \mu_T - 1.645\sigma_T$	$g = \mu_T + 1.645\sigma_T$	$g_l = \mu_T - 1.96\sigma_T$ $g_r = \mu_T + 1.96\sigma_T$
0.01	$g = \mu_T - 2.33\sigma_T$	$g = \mu_T + 2.33\sigma_T$	$g_l = \mu_T - 2.58\sigma_T$ $g_r = \mu_T + 2.58\sigma_T$

