CENTRALE COMMISSIE VOORTENTAMEN WISKUNDE

Entrance Exam Wiskunde A

Date: 22 July 2019
Time: 13.30 – 16.30 hours
Questions: 6

Please read the instructions below carefully before answering the questions. Failing to comply with these instructions may result in deduction of points.

Make sure your name is clearly written on every answer sheet.

Take a new answer sheet for every question.

Show all your calculations clearly. Illegible answers and answers without a calculation or an explanation of the use of your calculator are invalid (*see also question 1)*.

Write your answers in ink. Do not use a pencil, except when drawing graphs. Do not use correction fluid.

You can use a basic scientific calculator. **Other equipment, like a graphing calculator, a calculator with the option of computing integrals, a formula chart, BINAS or a book with tables, is NOT permitted.**

On the last two pages of this exam you will find a list of formulas.

You can use a dictionary if it is approved by the invigilator.

Please **switch off your mobile telephone** and put it in your bag.

Points that can be scored for each item:						
Question	1	2	3	4	5	6
a b c d	4 5 5 5	5 5 5	4 4 4 5	3 4 5 4	3 4 3 4	5 4
Total	19	15	17	16	14	9
$\text{Grade} = \frac{\text{total points scored}}{10} + 1$ You will pass the exam if your grade is at least 5.5 .						

Question 1 – Algebraic computations

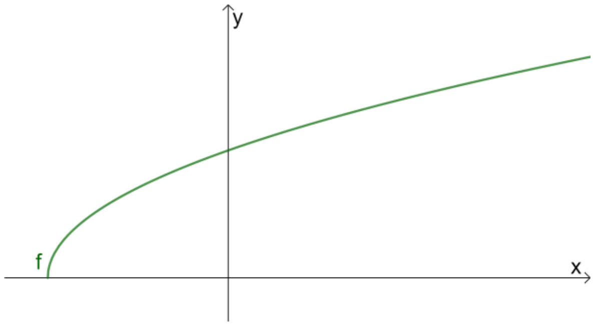
*When you are asked to perform a computation **algebraically**, your computation should be fully worked out on paper. Reading function values from a table (including tables produced by a calculator) is not allowed either in algebraic calculations. You can use a calculator for simple calculations and for approximations of numbers like* $\sqrt{2}$ *and* $\log(3)$.
Unless stated otherwise, all computations in this exam have to be performed algebraically.

4pt a Solve the equation $16x^3 + 9x = 24x^2$ algebraically.

Given are line *k* with equation $x - 4y = 2$ and line l with equation $y = 2x + 7$.
Line m is parallel to line *k* and passes through the origin $O(0,0)$.

5pt b Compute algebraically the coordinates of the intersection of lines l and m.

In the figure below the graph is shown of the function $f(x) = \sqrt{2x + 8}$.



A is the point on the graph of f where the slope of the graph equals $\frac{1}{4}$.

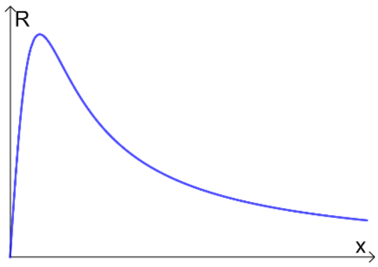
5pt c Use the derivative function of f to compute the coordinates of point *A*.

5pt d Compute algebraically the coordinates of the intersection(s) of the graph of f and the straight line with the equation $y = -2x + 4$.

Question 2 – Beautiful rain

Karen has a farm in Africa. The yield of a certain crop depends on the rainfall in the growing season, which is in the months of November and December. Too little or too much rainfall will lead to a lower yield. The yield of the crop in tons as a function of the amount of rainfall is given by the formula

$$R(x) = \frac{7500\,x}{x^2 + 2500}$$



In this formula, $R(x)$ is the yield of the crop in tons and x is the amount of rainfall in November and December, expressed in mm (1 mm rainfall = 1 liter per m^2).

The graph of this function is shown in the figure on the right.

In order to make a profit, the yield of the crop must be over 60 tons.

5pt **a** Compute algebraically the amount of rainfall at which the yield of the crop is more than 60 tons.

5pt **b** Compute algebraically the amount of rainfall at which the yield of the crop is maximal.

Denys also has a farm in Africa. The yield of his crop as a function of the amount of rainfall is given by

$$R(x) = 2x \cdot \mathrm{e}^{-0.02x}$$

($R(x)$ in tons, x in mm rainfall in the growing season).

Since there is a river nearby which provides plenty of water, Denys can artificially increase the rainfall by sprinkling.

5pt **c** Use the derivative of the yield function to determine whether it is a good idea to increase the rainfall artificially if the natural amount of rainfall in the growing season is 75 mm.

Question 3 – Wheel of fortune

To attract shoppers during the quiet summer season, the shops in a shopping mall give away vouchers to their customers. If you hand in such a voucher, you can give a spin to the wheel of fortune in the picture on the right, to win coupons for a reduction on the price of your next purchase in the shopping mall. First we assume that all 30 outcomes on this wheel of fortune are equally likely.

If the outcome is a vowel (A, E, I, O, U), you will get a coupon of 10 euros and if the outcome is one of the 21 consonants, you will get a coupon for 5 euros.

If the outcome is one of the four so called jokers, you must spin the wheel one more time. If the outcome of the second spin is a joker again, you will get a coupon of 100 euros, otherwise you will get a coupon of 5 euros, regardless of the type of letter of the outcome.

The value of the coupon that you can win in this way is a random variable X.

4pt a Show that $P(X=5)=\frac{734}{900}$ and $P(X=100)=\frac{16}{900}$.

The random variable X has an expectation of $E(X)=7.5222$ and a standard deviation of $\sigma(X)=12.5797$.

4pt b Give a clear computation to confirm that $E(X)=7.5222$.

At a certain day, this game is played 676 times. The total value of the coupons that are given away, is a random variable Y.

4pt c Compute $E(Y)$ and $\sigma(Y)$.

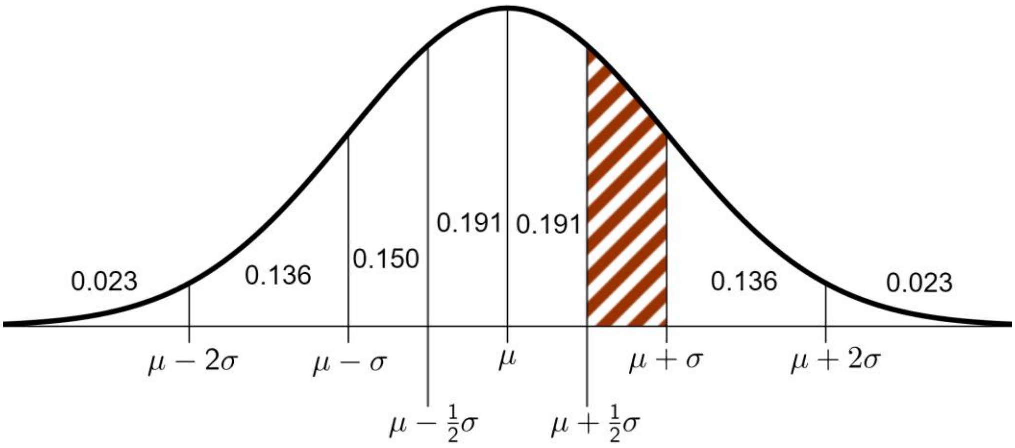
After a while, the owners of the shopping mall suspect that the probability that the outcome of a spin on the wheel of fortune is a joker, is higher than can be expected from the description given above. To test this, the outcomes of 100 spins of the wheel are noted.

5pt d Set up a statistical testing procedure for this situation by answering the following questions:
- What test statistic would you use?
- What kind of probability distribution does this test statistic have and what are its parameters?
- What is the null hypothesis and what is the alternative hypothesis?
- What additional information do you need to perform this testing procedure?

Question 4 – To fry or not to fry

Farmer Fred grows potatoes for a factory that produces French fries and mashed potato powder. Potatoes with a weight of more than 85 grams can be used for the production of French fries, potatoes with a lesser weight can only be used to produce mashed potato powder. Therefore, the factory pays 9 cents for each potato with a weight of more than 85 grams and it pays just 6 cents for a potato with a weight of less than 85 grams.
The weight of the potatoes in this year's harvest is normally distributed with a mean of $\mu = 83$ grams and a standard deviation of $\sigma = 4$ grams.

3pt a Use the figure below to show that the probability that a potato from this harvest weighs more than 85 grams, equals 0.309 .



A normal probability distribution X
The area of the shaded region represents $P\left(\mu + \frac{1}{2}\sigma < X < \mu + \sigma\right) = 0.150$

This year, Fred harvested 850 000 potatoes.

4pt b Compute the expected total price at which Fred will sell these potatoes to the factory.

At a quality control procedure, 50 potatoes are investigated which are randomly selected from this harvest.

5pt c Compute the probability that exactly 15 of these 50 potatoes have a weight of more than 85 grams.

Farmer Barney also grows potatoes for this factory. The weight of the potatoes in his harvest this year is normally distributed with a mean of $\mu = 82$ grams and a standard deviation of $\sigma = 7$ grams. Just as Fred, Barney harvested 850 000 potatoes.

4pt d Is the expected total price of Barney's harvest higher or lower than the expected total price of Fred's harvest? Explain your answer!

Question 5 – Interesting loans

Bert has bought a nice house in the middle of the Netherlands. To finance this house, he has taken out a mortgage and a bridging loan. The initial value of the mortgage is 126 000 euros. The interest rate is 2.4% per year and he pays back this mortgage in 30 years, in monthly instalments of $126\ 000/(30 \times 12) = 350$ euros. These instalments are due on the first of each month together with the interest, which is 0.2% of the remaining debt in the previous month. The transaction date of the sale of the house is 1 July 2019, so the first payment is due on 1 August 2019.

3pt a Copy the table below on your answer sheet and fill in the six missing entries.

Month	Remaining debt in euros	Interest due in euros
July 2019	126 000	--
August 2019	125 650	252
September 2019	125 300	
October 2019		
November 2019		
June 2049	350	
July 2049	0	0.70

4pt b Compute algebraically the total amount of interest Bert has to pay in these 30 years.

Bert has not sold his old house yet, so he also needs a bridging loan of 100 000 euros. During the bridging period, he does not pay any instalments nor does he pay any interest for this bridging loan. The full amount of the loan and the interest will be paid on the transaction date of the sale of his old house. This amount of course depends on the duration of the bridging period and is given by the formula

$$B(t) = 100\ 000 \cdot \mathrm{e}^{0.00015t}$$

In this formula, $B(t)$ is the total amount to be paid in euros and t is the time in days, with $t = 0$ on 1 July 2019.

Since most customers are not trained mathematicians, the bank that lends out this bridging loan cannot publish this formula. Instead, it publishes the so called effective yearly interest rate, which is the interest as a percentage of the debt that a customer would pay if he paid back this loan after exactly one year (= 365 days, the leap day is neglected).

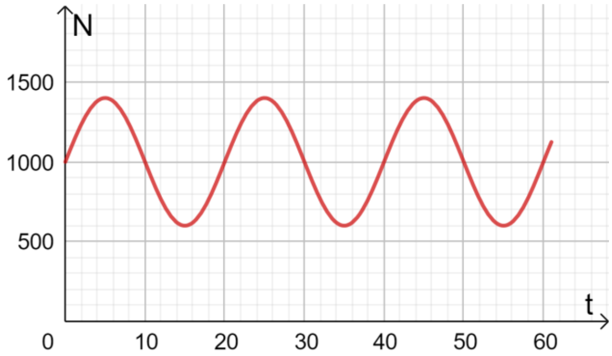
3pt c Compute algebraically the effective yearly interest rate for Bert's bridging loan.

After a while, Bert has sold his old house. The total amount that he has to pay for his bridging loan turns out to be 101 496 euros.

4pt d How many days did Bert need this bridging loan?

Question 6 – Field mice

The number of field mice in a meadow varies because of the prey-predator cycle. If the number of mice is low, there are not many predators, causing the mice population to grow. This will attract more predators which will in the end lead to an equally low number of mice, so the predators will leave and the mice population will grow again, etc. etc.
For a certain meadow, this phenomenon is modeled for the months July and August by the graph in the figure below.



The function that belongs to this graph has a formula of the form

$$N(t) = A + B\sin(Ct)$$

In this formula, $N(t)$ is the number of field mice in the meadow and t is the time in days, with $t = 0$ on 1 July.

5pt **a** Determine values of A, B and C that are in accordance with the information in the graph.

On 22 July, that is at $t = 21$, there are 1124 field mice in the meadow.

4pt **b** Compute the first two dates after 22 July at which there will be 1124 field mice in the meadow.

End of the exam.

Is your name on all answer sheets?

Formula list wiskunde A

Quadratic equations

The solutions of the equation $ax^2 + bx + c = 0$ with $a \neq 0$ and $b^2 - 4ac \geq 0$ are

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Differentiation

Rule	**function**	**derivative function**
Sum rule	$s(x) = f(x) + g(x)$	$s'(x) = f'(x) + g'(x)$
Product rule	$p(x) = f(x) \cdot g(x)$	$p'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
Quotient rule	$q(x) = \frac{f(x)}{g(x)}$	$q'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{\left(g(x)\right)^2}$
Chain rule	$k(x) = f(g(x))$	$k'(x) = f'\left(g(x)\right) \cdot g'(x)$ or $\frac{dk}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$

Logarithms

Rule	**conditions**
${}^{g}\log a + {}^{g}\log b = {}^{g}\log ab$	$g > 0, \quad g \neq 1, \quad a > 0, \quad b > 0$
${}^{g}\log a - {}^{g}\log b = {}^{g}\log \frac{a}{b}$	$g > 0, \quad g \neq 1, \quad a > 0, \quad b > 0$
${}^{g}\log a^p = p \cdot {}^{g}\log a$	$g > 0, \quad g \neq 1, \quad a > 0$
${}^{g}\log a = \frac{{}^{p}\log a}{{}^{p}\log g}$	$g > 0, \quad g \neq 1, \quad a > 0, \quad p > 0, \quad p \neq 1$

Arithmetic and geometric sequences

Arithmetic sequence:	$Sum = \frac{1}{2} \cdot number\ of\ terms \cdot (u_e + u_l)$
Geometric sequence:	$Sum = \frac{u_{l+1} - u_e}{r - 1} \quad (r \neq 1)$
In both formulas:	*e* = number first term of the sum; *l* = number last term of the sum

More formulas on the next page.

Formula list wiskunde A (continued)

Probability

If *X* and *Y* are any random variables, then: $E(X+Y)=E(X)+E(Y)$
If furthermore *X* and *Y* are independent, then: $\sigma(X+Y)=\sqrt{\sigma^2(X)+\sigma^2(Y)}$

$\sqrt{n}$-law:
For *n* independent repetitions of the same experiment where the result of each experiment is a random variable *X*, the sum of the results is a random variable *S* and the mean of the results is a random variable $\overline{X}$.

$E(S)=n\cdot E(X)$ $\qquad$ $\sigma(S)=\sqrt{n}\cdot\sigma(X)$

$E(\overline{X})=E(X)$ $\qquad$ $\sigma(\overline{X})=\dfrac{\sigma(X)}{\sqrt{n}}$

Binomial Distribution

If *X* has a binomial distribution with parameters *n* (number of experiments) and *p* (probability of success at each experiment), then

$$P(X=k)=\binom{n}{k}\cdot p^k\cdot(1-p)^{n-k}\quad \text{with}\quad k=0,1,2,\ldots,n$$

Expected value: $E(X)=np$ $\qquad$ Standard deviation: $\sigma(X)=\sqrt{n\cdot p\cdot(1-p)}$

n and *p* are the parameters of the binomial distribution

Normal Distribution

If *X* is a normally distributed random variable with mean μ and standard deviation σ, then

$Z=\frac{X-\mu}{\sigma}$ has a standard normal distribution and $P(X<g)=P\left(Z<\frac{g-\mu}{\sigma}\right)$

μ and σ are the parameters of the normal distribution.

Hypothesis testing

In a testing procedure where the test statistic *T* is normally distributed with mean μ_T standard deviation σ_T the boundaries of the rejection region (the critical region) are::

α	left sided	right sided	two sided
0.05	$g=\mu_T-1.645\sigma_T$	$g=\mu_T+1.645\sigma_T$	$g_l=\mu_T-1.96\sigma_T$
			$g_r=\mu_T+1.96\sigma_T$
0.01	$g=\mu_T-2.33\sigma_T$	$g=\mu_T+2.33\sigma_T$	$g_l=\mu_T-2.58\sigma_T$
			$g_r=\mu_T+2.58\sigma_T$

Answers

1a $x = 0 \vee x = \frac{3}{4}$

1b $x = -4;\ y = -1$

1c $x = 4;\ y = 4$

1d $x = \frac{1}{2};\ y = 3$

2a $25 < x < 100$

2b $x = 50$

2c $R'(75) = -\mathrm{e}^{-1.5} < 0$
Increasing the rainfall artificially will thus decrease the yield, so that is not a good idea

3a $P(X = 100) = \frac{4}{30} \cdot \frac{4}{30} = \frac{16}{900}$;
$P(X = 5) = \frac{21}{30} + \frac{4}{30} \cdot \frac{26}{30} = \frac{734}{900}$ or $P(X = 5) = 1 - \big(P(X = 100) + P(X = 10)\big) = \frac{734}{900}$

3b $E(X) = 5 \cdot P(X = 5) + 10 \cdot P(X = 10) + 100 \cdot P(X = 100) = 5 \cdot \frac{734}{900} + 10 \cdot \frac{5}{30} + 100 \cdot \frac{16}{900}$

3c $E(Y) = 676 \cdot E(X) = 676 \cdot 7.5222 = 5085.0;\ \sigma(Y) = \sqrt{676} \cdot \sigma(X) = 26 \cdot 12.5797 = 327.1$

3d The test statistic in the number of times that a joker is drawn.
The test statistic has a binomial distribution with $n = 100$ and $p = \frac{4}{30}$
$H_0\colon\ p = \frac{4}{30};\ H_1\colon\ p > \frac{4}{30}$
You also need the significance level α and the result of the sample

4a $85 = 83 + 2 = \mu + \frac{1}{2}\sigma$
We are looking for the fraction of the potatoes with a weight over $\mu + \frac{1}{2}\sigma$
This is $0.150 + 0.136 + 0.023$

4b 5 887 950 cent

4c $\binom{50}{15} \cdot 0.309^{15} \cdot (1 - 0.309)^{35} = 0.121186$

4d Barney's total price is higher than Fred's if a larger fraction of his potatoes weighs over 85 grams. For Barney we have $\mu + \frac{1}{2}\sigma = 82 + 3\frac{1}{2} = 85\frac{1}{2}$, so the probability that one of Barney's potatoes weighs over 85 grams is larger than 0.309.
Therefore, Barney's total price is (expected to be) higher

5a Remaining debt 1-10-2019: 124 950; 1-11-2019: 124 600
Interest 1-9-2019: 251.30; 1-10-2019: 250.60; 1-11-2019: 249.90; 1-6-2049: 1.40

5b $\frac{1}{2} \cdot 360 \cdot (252 + 0{,}70) = 45\,486$

5c 5.62765%

5d 99 days

6a $A = 1000;\ B = 400;\ C = \frac{2\pi}{20}$

6b 30 July and 11 August

