CENTRALE COMMISSIE VOORTENTAMEN WISKUNDE

Entrance Exam Wiskunde A

Date: 18 December 2023
Time: 13.30 – 16.30
Questions: 6

Please read the instructions below carefully before answering the questions. Failing to comply with these instructions may result in deduction of points.

Make sure your name is clearly written on every answer sheet.

Take a new answer sheet for every question.

Show all your calculations clearly. Illegible answers and answers without a calculation or an explanation of the use of your calculator are invalid (*see also question 1)*.

Write your answers in ink. Do not use a pencil, except when drawing graphs. Do not use correction fluid.

You can use a basic scientific calculator. **Other equipment, like a graphing calculator, a calculator with the option of computing integrals, a formula chart, BINAS or a book with tables, is NOT permitted.**

On the last two pages of this exam you will find a list of formulas.

You can use a dictionary if it is approved by the invigilator.

Please **switch off your mobile telephone** and put it in your bag.

Points that can be scored for each item:						
Question	1	2	3	4	5	6
a b c e	6 6 5	3 8	7 5 5 4	2 4 2	4 4	5 3 6 2
Total	17	11	21	8	8	16
$\text{Grade} = \dfrac{\text{total points scored}}{9} + 1$ You will pass the exam if your grade is at least 5.5 .						

Question 1 – Algebraic computations

Take a new answer sheet for every question!

When you are asked to perform a computation ***algebraically****, your computation should be fully worked out on paper. Reading function values from a table (including tables produced by a calculator) is not allowed in algebraic calculations. You can use a calculator for simple calculations and for approximations of numbers like* $\sqrt{2}$ *and* $\log(3)$.
Unless stated otherwise, all computations in this exam have to be performed algebraically.

The function f is given by $f(x) = 4x^3 + 9x^2 - 12x$.
The graph of f has a minimum in point A and a maximum in point B.

6pt a Compute algebraically the slope of the straight line through point A and point B.

The function g is given by $g(x) = \frac{x-2}{3x-2}$.
Points C and D are the points on the graph of g at which the tangent line to this graph is parallel to the line with equation $y = x$.

6pt b Compute algebraically the *x*-coordinates of points C and D.

The relationship between the quantities P and Q is given by the formula

$$\log(P) = 4 + 2 \cdot \log(Q)$$

This relationship can also be expressed in a formula of the form

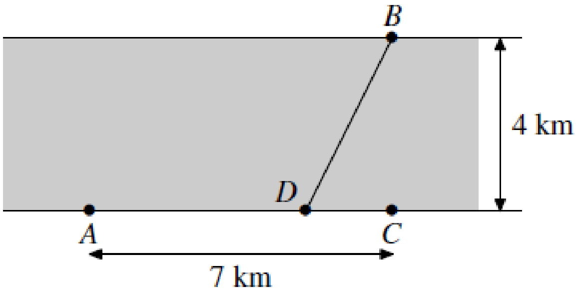
$$Q = a \cdot P^b$$

5pt c Compute algebraically the values of a and b in this second formula.

Question 2 – Cycling from A to B

Take a new answer sheet for every question!

The county of Cyclalot wants to build a cycle path from A to B through a nature reserve (shaded in the figure below).



The costs for building a cycle path on the edge of the nature reserve (from point A in the direction of point C, which is directly below point B) are 30 000 euro per kilometer. The costs for building a cycle path through the nature reserve are 50 000 euro per kilometer. The county is looking for a point D between A and C such that the costs for building a cycle path from A via D to B are minimal.

If we denote the distance in kilometers between A and D by x, the distance in kilometers between D and B is given by $\sqrt{x^2 - 14x + 65}$.

3pt a Use the Pythagorean Theorem to prove this.

The total costs for the construction the cycle path from A via D to B in tens of thousands of euros are given by

$$TC = 3x + 5\sqrt{x^2 - 14x + 65}$$

8pt b Compute algebraically the value of x for which TC is minimal.

Question 3 – A bag with clean socks

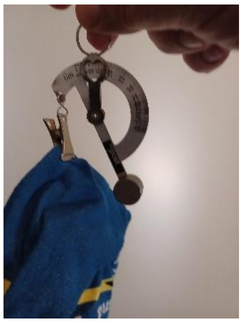
Take a new answer sheet for every question!

After doing the laundry, George puts his clean socks in a bag. At a certain moment, this bag contains 4 red socks, 4 yellow socks and 2 blue socks. George takes socks from this bag (randomly and without replacement) until he has two socks of the same colour. The number of socks that he takes from the bag is a random variable X.
If George takes the socks in the order red, yellow, blue, red, he has 2 socks of the same colour and thus we have $X = 4$.

For this random variable we have $P(X = 2) = \frac{13}{45}$ and $P(X = 4) = \frac{12}{45}$.

7pt a Show that this is true.

5pt b Compute $E(X)$.



On a rainy day, George measures the weight of his socks.
His conclusion is that this weight is normally distributed with an average of $\mu = 13$ g and a standard deviation of $\sigma = 0.5$ g.
In the remainder of this question and in question 4 you may assume that this conclusion is right.

At a certain moment, George decides to do something more useful and he puts his laundry in the washing machine. Amongst many other items, there are 13 pairs of socks (= 26 socks) in his laundry.

5pt c According to the rules of thumb, how many of these 26 socks have a weight between 12.5 g and 14 g?

4pt d Compute the probability that exactly 5 of the 10 socks that were originally in the bag weigh more than 13 g.

Question 4 – Buying new socks

Take a new answer sheet for every question!

George wants to buy new socks. Before he buys a large number of socks, he wants to test whether the weight of the socks in the shop is the same as the weight of the socks that he has measured in question 3, where he found $\mu = 13$. To test this, he buys 8 pairs of socks (= 16 socks). In this testing procedure, he assumes that the weight of the socks in the shop is normally distributed with a standard deviation of $\sigma = 0.5$ g and he takes a significance level of $\alpha = 0.05$.

2pt a State the null hypothesis and the alternative hypothesis for this test procedure.

The average weight of the 16 socks is 12.77 g. This yields a *p*-value of 0.033.

4pt b Compute the average and the standard deviation of the test statistic that is used to calculate this *p*-value.

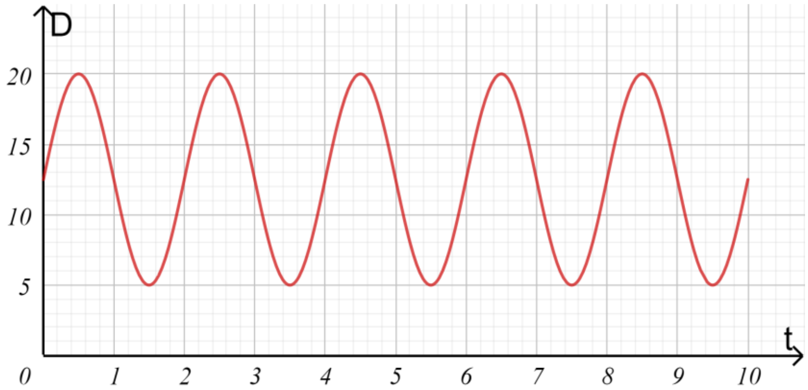
2pt c What is the conclusion of this test procedure?
Explain your answer!

Question 5 – Pistons in cylinders

Take a new answer sheet for every question!

The cylinder is an important part of a petrol engine. In a cylinder a piston moves up and down. In this way, the energy released when burning petrol at the top of the cylinder is converted into a rotating movement of the rod at the bottom of the cylinder.
A petrol car engine usually contains 4 or 8 cylinders.

The figure below shows the relationship between the distance from the top of a certain type of cylinder and the top of the piston in centimetres (D) and the time in seconds (t).



This relationship can be represented by a formula of the form $D = a + b\sin(ct)$.

4pt a Determine the values of a, b and c in this formula.

For another type of cylinder, the relationship between the distance between the top of the cylinder and the top of the piston in centimetres (D) and the time in seconds (t) is given by

$$D = 20 + 10\sin\left(\frac{2}{3}\pi t - \frac{3}{4}\pi\right)$$

4pt b Compute algebraically the first three times after $t = 0$ at which D is minimal according to this formula.

Question 6 – The growth of two cities

Take a new answer sheet for every question!

The population of city *A* grew exponentially between 1950 and 2000.
The growth percentage was 44% in 10 years.

5pt a Compute algebraically the time in months in which the population of city *A* doubled between 1950 and 2000.

3pt b Compute algebraically the percentage at which the population of city *A* grew between 1 January 1985 and 1 January 1990.

The population of city *B* is given by the formula

$$N = 20 \cdot (25 - 20\mathrm{e}^{-0.2t}).$$

In this formula, N is the population of city *B* in thousands and t is the time in years, with $t = 0$ on 1 January 1950.

6pt c Compute algebraically the time (year and month) at which the population of city *B* was equal to 200 000.

2pt d Compute the maximal population of city *B* according to the formula given above.

End of the exam.

*When you have finished the exam, check whether your **name** and the **question number** are on every answer sheet.*

Place the answer sheets in the correct order in the plastic folder and place the sheet with your data in the front in this folder.

*What should **not** be in the folder:*
- empty sheets, please leave them on your table;
- sheets with only your name on it, please take them with you;
- scrap paper;
- these questions.

This is the only way we can ensure a smooth correction of your exam work.

Remain seated until one of the invigilators collects your folder (or calls you).

Formula list wiskunde A

Quadratic equations

The solutions of the equation $ax^2 + bx + c = 0$ with $a \neq 0$ and $b^2 - 4ac \geq 0$ are

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Differentiation

Rule	**function**	**derivative function**
Sum rule	$s(x) = f(x) + g(x)$	$s'(x) = f'(x) + g'(x)$
Product rule	$p(x) = f(x) \cdot g(x)$	$p'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
Quotient rule	$q(x) = \frac{f(x)}{g(x)}$	$q'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{\left(g(x)\right)^2}$
Chain rule	$k(x) = f(g(x))$	$k'(x) = f'\left(g(x)\right) \cdot g'(x)$ or $\frac{dk}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$

Logarithms

Rule	**conditions**
${}^{g}\log a + {}^{g}\log b = {}^{g}\log ab$	$g > 0, \quad g \neq 1, \quad a > 0, \quad b > 0$
${}^{g}\log a - {}^{g}\log b = {}^{g}\log \frac{a}{b}$	$g > 0, \quad g \neq 1, \quad a > 0, \quad b > 0$
${}^{g}\log a^p = p \cdot {}^{g}\log a$	$g > 0, \quad g \neq 1, \quad a > 0$
${}^{g}\log a = \frac{{}^{p}\log a}{{}^{p}\log g}$	$g > 0, \quad g \neq 1, \quad a > 0, \quad p > 0, \quad p \neq 1$

Arithmetic and geometric sequences

Arithmetic sequence:	$Sum = \frac{1}{2} \cdot number\ of\ terms \cdot (u_e + u_l)$
Geometric sequence:	$Sum = \frac{u_{l+1} - u_e}{r - 1} \quad (r \neq 1)$
In both formulas:	*e* = number first term of the sum; *l* = number last term of the sum

More formulas on the next page.

Formula list wiskunde A (continued)

Probability

If X and Y are any random variables, then: $E(X+Y) = E(X) + E(Y)$
If furthermore X and Y are independent, then: $\sigma(X+Y) = \sqrt{\sigma^2(X) + \sigma^2(Y)}$

$\sqrt{n}$-law:
For n independent repetitions of the same experiment where the result of each experiment is a random variable X, the sum of the results is a random variable S and the mean of the results is a random variable $\overline{X}$.

$E(S) = n \cdot E(X)$ $\quad\quad$ $\sigma(S) = \sqrt{n} \cdot \sigma(X)$

$E(\overline{X}) = E(X)$ $\quad\quad$ $\sigma(\overline{X}) = \dfrac{\sigma(X)}{\sqrt{n}}$

Binomial Distribution

If X has a binomial distribution with parameters n (number of experiments) and p (probability of success at each experiment), then

$$P(X = k) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} \quad \text{with} \quad k = 0, 1, 2, \ldots, n$$

Expected value: $E(X) = np$ $\quad\quad$ Standard deviation: $\sigma(X) = \sqrt{n \cdot p \cdot (1-p)}$

n and p are the parameters of the binomial distribution

Normal Distribution

If X is a normally distributed random variable with mean μ and standard deviation σ, then

$Z = \frac{X-\mu}{\sigma}$ has a standard normal distribution and $P(X < g) = P\left(Z < \frac{g-\mu}{\sigma}\right)$

μ and σ are the parameters of the normal distribution.

Hypothesis testing

In a testing procedure where the test statistic T is normally distributed with mean μ_T standard deviation σ_T the boundaries of the rejection region (the critical region) are:

α	left sided	right sided	two sided
0.05	$g = \mu_T - 1.645\sigma_T$	$g = \mu_T + 1.645\sigma_T$	$g_l = \mu_T - 1.96\sigma_T$
			$g_r = \mu_T + 1.96\sigma_T$
0.01	$g = \mu_T - 2.33\sigma_T$	$g = \mu_T + 2.33\sigma_T$	$g_l = \mu_T - 2.58\sigma_T$
			$g_r = \mu_T + 2.58\sigma_T$

