CENTRALE COMMISSIE VOORTENTAMEN WISKUNDE

Entrance Exam Wiskunde A

Date: 15 December 2022
Time: 13.30 – 16.30
Questions: 6

Please read the instructions below carefully before answering the questions. Failing to comply with these instructions may result in deduction of points.

Make sure your name is clearly written on every answer sheet.

Take a new answer sheet for every question.

Show all your calculations clearly. Illegible answers and answers without a calculation or an explanation of the use of your calculator are invalid (*see also question 1)*.

Write your answers in ink. Do not use a pencil, except when drawing graphs. Do not use correction fluid.

You can use a basic scientific calculator. **Other equipment, like a graphing calculator, a calculator with the option of computing integrals, a formula chart, BINAS or a book with tables, is NOT permitted.**

On the last two pages of this exam you will find a list of formulas.

You can use a dictionary if it is approved by the invigilator.

Please **switch off your mobile telephone** and put it in your bag.

Points that can be scored for each item:						
Question	1	2	3	4	5	6
a b c d e	6 5 4 4	2 4 4	4 5 2 4 2	4 5 5	2 6	2 6 5
Total	19	10	17	14	8	13
$\text{Grade} = \frac{\text{total points scored}}{9} + 1$ You will pass the exam if your grade is at least 5.5 .						

Question 1 – Algebraic computations

Take a new answer sheet for every question!

*When you are asked to perform a computation **algebraically**, your computation should be fully worked out on paper. Reading function values from a table (including tables produced by a calculator) is not allowed in algebraic calculations. You can use a calculator for simple calculations and for approximations of numbers like $\sqrt{2}$ and* $\log(3)$.
Unless stated otherwise, all computations in this exam have to be performed algebraically.

The function f is given by $f(x) = 48x^4 - 64x^3 + 24x^2 - 1$.

6pt a Compute algebraically the minimal value of $f(x)$.

The function g is given by $g(x) = x \cdot \mathrm{e}^{-2x}$.

5pt b Compute algebraically an equation for the tangent line to the graph of g in the origin $O(0,0)$.

The function h is given by $h(x) = 3 \cdot 4^x$.

4pt c Solve the equation $h(x) = 108$ algebraically.
Give an approximation of the answer rounded to 3 digits behind the decimal point.

The relationship between the quantities Q and R is given by the formula

$$\log(R) = \frac{1}{5}Q - 3$$

This formula can be transformed into a formula of the form

$$R = c \cdot d^Q$$

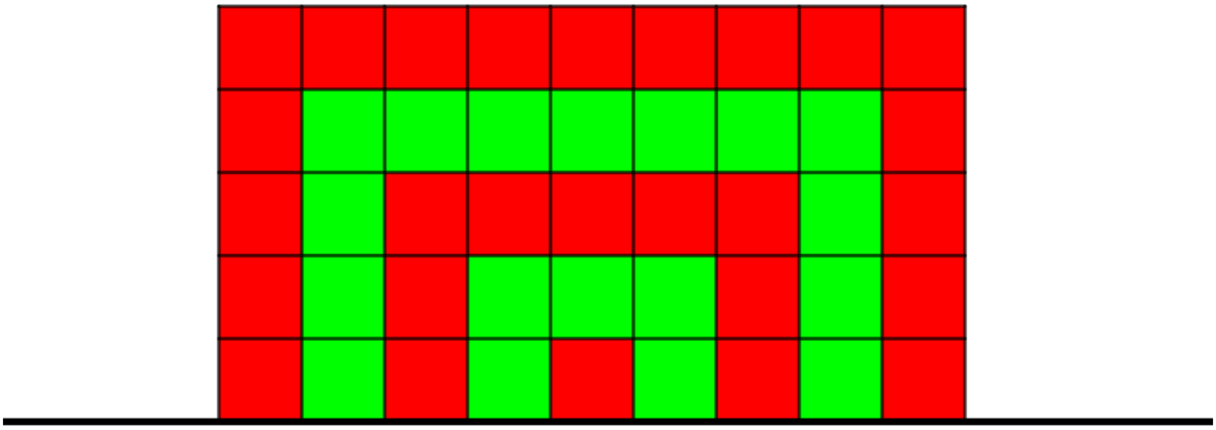
4pt d Compute algebraically the values of c and d in this second formula.

Question 2 – Tiles on a patio

Take a new answer sheet for every question!

Bert is laying tiles on his patio.
In the first step, he puts a red tile at the edge of the patio.
In the second step, he adds a green tile to the left and to the right and a row of green tiles on top of the first three tiles.
In the next steps, he continues adding red and green tiles in a similar way.

In the figure below, the pattern of the tiles after five steps is shown.



2pt **a** How many green tiles are added in the sixth step?

The number of tiles that is added in step n is given by the recursive formula

$$\begin{cases} a(n+1) = a(n) + 4 \\ a(1) = 1 \end{cases}$$

The direct formula for this number is $a(n) = 4n - 3$.

4pt **b** Show how this direct formula is derived from the recursive formula.

The total number of tiles that is laid after step n, is given by a formula of the form $t(n) = pn^2 - qn$.

4pt **c** Use the sum formula for an arithmetic sequence to compute the values of p and q in this formula.

Question 3 – Fruit and a raven

Take a new answer sheet for every question!

The children's game of *Orchid* is about a fruit basket with four colours of fruit, and a raven, that tries to steal the fruit basket.
At the start of a turn, the player has to throw a die which has four coloured sides (green, blue, red, and yellow), one side with the raven and one side with a fruit basket, If one of the colours comes up, one fruit of that colour is put in the fruit basket and if the side with the fruit basket comes up, the player can put a fruit of any colour in the fruit basket. However, if the side with the raven comes up, the raven gets one step closer to stealing the fruit basket.

In questions a and b, assume that the die is fair, that is that each of the six sides has an equal probability of coming up at each throw.

Hielke plays this game with his grandchildren. They notice that in the first 40 turns, the side with the raven never comes up. After some calculating, Hielke assumes that the probability of this happening is less than 1 in 1500.

4pt a Investigate whether Hielke's assumption is right.

5pt b Compute the probability that the side with the fruit basket comes up at least two times in the first ten throws of a game.

After playing this game a number of times, Hielke suspects that the die is not fair. He decides to test this by throwing the die 100 times and counting the number of times that the side with the raven comes up in these 100 throws.

2pt c State the null hypothesis and the alternative hypothesis for this test procedure.

In the 100 throws, the side with the raven comes up 10 times. This results in a p-value of 0.0477.

4pt d Compute the expectation and the standard deviation of the test statistic that is used to find this p-value.

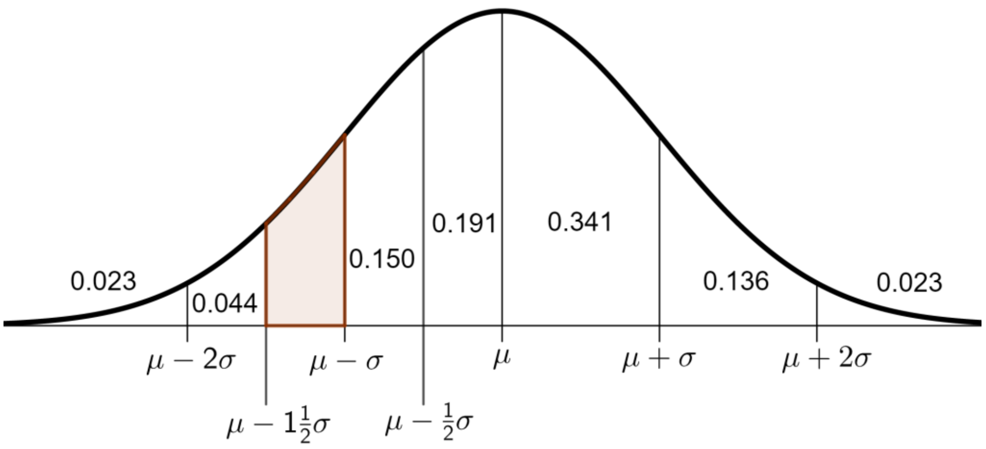
2pt e Can you draw a conclusion from this testing procedure?
If so, what is this conclusion and why?
If not, point out all the information that you furthermore need to be able to draw a conclusion.

Question 4 – Apples in a box

Take a new answer sheet for every question!

The weight of the famous Dutch *Elstar* apples is normally distributed with an average of $\mu = 145$ g and a standard deviation of $\sigma = 10$ g.

4pt a Use the figure below to compute the percentage of these apples that have a weight between 145 g and 160 g.



A normal probability distribution X
The area of the shaded region represents $P\left(\mu - 1\frac{1}{2}\sigma < X < \mu - \sigma\right) = 0.092$

A supermarket sells these apples in boxes containing four apples each. The weight of the empty boxes is normally distributed with an average of $\mu = 20$ g and a standard deviation of $\sigma = 1$ g.

5pt b Compute the average and the standard deviation of the total weight (apples + empty box) of such a box.

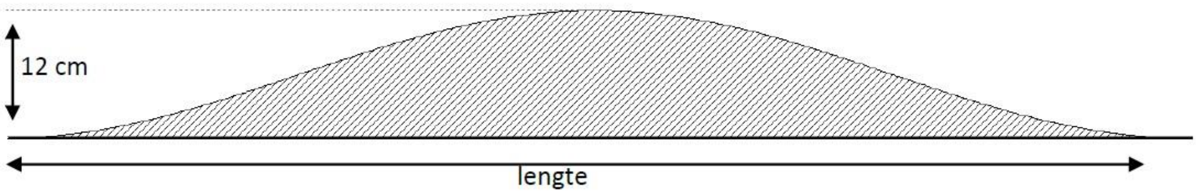
Saskia buys three of these boxes and weighs the twelve apples in these boxes. Five of these apples weigh more than 150 g. After weighing, Saskia puts the apples in a fruit basket.
Peter randomly picks three apples from this fruit basket.

5pt c Compute the probability that exactly two of the apples that Peter picks weigh more than 150 g.

Question 5 – Speed bumps

Take a new answer sheet for every question!

In Belgium, the shape and dimensions of speed bumps have been laid down by law since 1983. The side view of a speed bump has the shape of a full sine wave. See the figure below.



lengte = length

For the speed bump in the figure above, which corresponds to a maximum speed of 30 km/h, the following formula has been drawn up:

$$h = 0.06 + 0.06 \cdot \sin\left(\frac{1}{2}\pi x - \frac{1}{2}\pi\right)$$

In this formula, h is the height and x is the horizontal distance from the left end of the speed bump, both in meters.

2pt **a** Compute the length of this speed bump algebraically.

A speed bump that corresponds to a maximum speed of 60 km/h is 12 meters long and 14 cm high. This leads to a formula of the form

$$h = a + b \cdot \sin\big(c(x - d)\big)$$

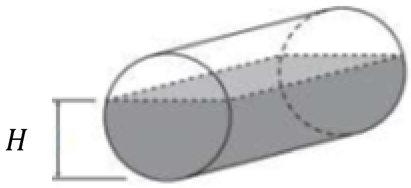
In this formula, h is again the height and x is again the horizontal distance from the left end of the speed bump, both in meters.

6pt **b** Find the values of a, b, c and d in this formula.
Explain your answers.

Question 6 – Emptying barrel

Take a new answer sheet for every question!

The figure below shows a horizontal barrel that is partially filled with a liquid. The circular front and back have a diameter of 1.8 meters.



The barrel is emptied from the given situation.
The height in meters H of the liquid level t minutes after the start of the emptying is given by the formula

$$H = 1.8 - (0.216 + 0.0039 \cdot t)^{\frac{2}{3}}$$

2pt **a** Compute algebraically the height of the liquid level one half hour after the start of the emptying.

At a certain moment, the barrel is half full.

6pt **b** Compute algebraically the time at this moment.
Give your answer in seconds accurately.

5pt **c** Use the derivative $\frac{\mathrm{d}H}{\mathrm{d}t}$ to compute algebraically the speed in centimeters per minute at which the liquid level in the barrel drops at $t = 10$.

End of the exam.

*When you have finished the exam, check whether your **name** and the **question number** are on every answer sheet.*

Place the answer sheets in the correct order in the plastic folder and place the sheet with your data in the front in this folder.

*What should **not** be in the folder:*
- empty sheets, please leave them on your table;
- sheets with only your name on it, please take them with you;
- scrap paper;
- these questions.

This is the only way we can ensure a smooth correction of your exam work.

Remain seated until one of the invigilators collects your folder (or calls you).

Formula list wiskunde A

Quadratic equations

The solutions of the equation $ax^2 + bx + c = 0$ with $a \neq 0$ and $b^2 - 4ac \geq 0$ are

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Differentiation

Rule	**function**	**derivative function**
Sum rule	$s(x) = f(x) + g(x)$	$s'(x) = f'(x) + g'(x)$
Product rule	$p(x) = f(x) \cdot g(x)$	$p'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
Quotient rule	$q(x) = \frac{f(x)}{g(x)}$	$q'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{\left(g(x)\right)^2}$
Chain rule	$k(x) = f(g(x))$	$k'(x) = f'\left(g(x)\right) \cdot g'(x)$ or $\frac{dk}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$

Logarithms

Rule	**conditions**
${}^{g}\log a + {}^{g}\log b = {}^{g}\log ab$	$g > 0,\ g \neq 1,\ a > 0,\ b > 0$
${}^{g}\log a - {}^{g}\log b = {}^{g}\log \frac{a}{b}$	$g > 0,\ g \neq 1,\ a > 0,\ b > 0$
${}^{g}\log a^p = p \cdot {}^{g}\log a$	$g > 0,\ g \neq 1,\ a > 0$
${}^{g}\log a = \frac{{}^{p}\log a}{{}^{p}\log g}$	$g > 0,\ g \neq 1,\ a > 0,\ p > 0,\ p \neq 1$

Arithmetic and geometric sequences

Arithmetic sequence:	$Sum = \frac{1}{2} \cdot number\ of\ terms \cdot (u_e + u_l)$
Geometric sequence:	$Sum = \frac{u_{l+1} - u_e}{r - 1} \quad (r \neq 1)$
In both formulas:	*e* = number first term of the sum; *l* = number last term of the sum

More formulas on the next page.

Formula list wiskunde A (continued)

Probability

If *X* and *Y* are any random variables, then: $E(X+Y) = E(X) + E(Y)$
If furthermore *X* and *Y* are independent, then: $\sigma(X+Y) = \sqrt{\sigma^2(X) + \sigma^2(Y)}$

$\sqrt{n}$-law:
For *n* independent repetitions of the same experiment where the result of each experiment is a random variable *X*, the sum of the results is a random variable *S* and the mean of the results is a random variable $\overline{X}$.

$E(S) = n \cdot E(X)$ $\qquad$ $\sigma(S) = \sqrt{n} \cdot \sigma(X)$

$E(\overline{X}) = E(X)$ $\qquad$ $\sigma(\overline{X}) = \dfrac{\sigma(X)}{\sqrt{n}}$

Binomial Distribution

If *X* has a binomial distribution with parameters *n* (number of experiments) and *p* (probability of success at each experiment), then

$$P(X = k) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} \quad \text{with} \quad k = 0, 1, 2, \ldots, n$$

Expected value: $E(X) = np$ $\qquad$ Standard deviation: $\sigma(X) = \sqrt{n \cdot p \cdot (1-p)}$

n and *p* are the parameters of the binomial distribution

Normal Distribution

If *X* is a normally distributed random variable with mean μ and standard deviation σ, then

$Z = \frac{X-\mu}{\sigma}$ has a standard normal distribution and $P(X < g) = P\left(Z < \frac{g-\mu}{\sigma}\right)$

μ and σ are the parameters of the normal distribution.

Hypothesis testing

In a testing procedure where the test statistic *T* is normally distributed with mean μ_T standard deviation σ_T the boundaries of the rejection region (the critical region) are:

α	left sided	right sided	two sided
0.05	$g = \mu_T - 1.645\sigma_T$	$g = \mu_T + 1.645\sigma_T$	$g_l = \mu_T - 1.96\sigma_T$
			$g_r = \mu_T + 1.96\sigma_T$
0.01	$g = \mu_T - 2.33\sigma_T$	$g = \mu_T + 2.33\sigma_T$	$g_l = \mu_T - 2.58\sigma_T$
			$g_r = \mu_T + 2.58\sigma_T$

